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Downside Risk Neutral Probabilities

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Abstract:
The price of any asset can be expressed with risk neutral probabilities, which are adjusted to incorporate risk preferences. This paper introduces the concepts of downside (respectively outer) risk neutral probabilities, which are adjusted to incorporate the preferences for downside (resp. outer) risk and higher degree risks. We derive new asset pricing formulas that rely on these probability measures. Downside risk neutral probabilities allow to value assets in a simple mean-variance framework. The associated pricing kernel is linear in wealth, as in the CAPM. With outer risk neutral probabilities, the pricing kernel is quadratic in wealth, and can be U-shaped.

Keywords: asset pricing, downside risk, quadratic pricing kernel, linear pricing kernel, prudence, risk neutral probabilities.

JEL Classification: D81, G12

How can risk preferences be incorporated into an asset pricing formula? One approach is to work directly with the utility function u, and to use the stochastic discount factor formula. The price of an asset is then the expectation of its payoff, as weighted state-by-state by the relative marginal utility of wealth in this state. Another (equivalent) approach is to adjust the probabilities of states of the world in such a way that the asset price is simply equal to the discounted expected payoff of the asset under the risk neutral measure. Thus, "risk neutral probabilities" allow to value assets in a simple risk neutral framework.[1]

Risk neutral probabilities are adjusted to take into account risk preferences. This includes risk aversion, which is an aversion to the dispersion of a distribution in the sense of mean-preserving spreads, but also higher order risk attitudes, including the aversion to downside risk and the aversion to outer risk.[2] A distribution has more downside risk if risk is transferred to the left of the distribution in a way which leaves the mean and variance unchanged. An increase in downside risk implies a lower third moment of the distribution, i.e., a lower skewness (Menezes, Geiss, and Tressler (1980), Chiu (2005)). In an expected utility framework, aversion to downside risk or "prudence" is equivalent to $u''' > 0$ (Menezes, Geiss, and Tressler (1980)), which is necessary for decreasing absolute risk aversion, for "standard risk aversion" (Kimball (1993)), and has been linked with precautionary savings (Kimball (1990)). A distribution has more outer risk if probability mass is transferred from the center toward the tails in a way which leaves the mean, variance, and skewness unchanged. Aversion to outer risk or "temperance" is equivalent to $u^{(4)} < 0$ (Menezes and Wang (2005)), and implies an aversion to the kurtosis of the distribution (Crainich and Eeckhoudt (2011)). These higher-order risk preferences can thus explain why especially the skewness but also the kurtosis of returns are determinants of expected returns (e.g., Harvey and Siddique (2000), Dittmar (2002)). Note that CARA and CRRA utility functions which are risk averse are also such that $u''' > 0$ and $u^{(4)} < 0$, and are therefore averse to downside risk and to outer risk.

This paper introduces downside (respectively outer) risk neutral probabilities, which are

[1]There are many variations on these two approaches. For example, assets can also be valued using state prices, where each state price is simply equal to the probability of the state multiplied by the stochastic discount factor in this state.

[2]In an expected utility framework, the signs of successive derivatives of the utility function can be directly related to the preferences for successive degrees of risk, but they cannot be directly related to preferences for moments of the distribution. Indeed, stochastic dominance criteria are related to degrees of risk rather than to moments of the distribution. For example, considering two distributions A and B with the same mean, distribution B is dominated in a second-order stochastic dominance sense if and only if it can be constructed by applying a sequence of mean-preserving spreads to distribution A (e.g. Gollier (2001)). Moreover, a change in a degree of risk implies a certain change in the corresponding moment of the distribution, but the opposite is not true. For example, a mean-preserving spread implies a higher variance, but a higher variance does not imply a mean-preserving spread of the distribution. That is, if two distributions have the same mean but different variances, the distribution with the lower variance will not necessarily be preferred by a risk averse agent. Thus, to study the effect of risk preferences on asset prices in an expected utility framework, we work with degrees of risk rather than with moments.

adjusted to take into account the aversion to downside (resp. outer) risk and higher-order risk attitudes, and are constructed based on a first (resp. second) order Taylor expansion of marginal utility.[3] This is by contrast with risk neutral probabilities, which are adjusted according to risk aversion and higher-order risk attitudes. Risk neutral probabilities allow to price assets "as if" investors were risk neutral – in the sense that, if investors were risk neutral, the corresponding asset pricing formula with *physical* probabilities would be used to price assets. Likewise, downside risk neutral probabilities allow to price assets "as if" investors were averse to risk, but were neutral with respect to higher degree risks, including downside risk and outer risk – if investors were downside risk neutral, the corresponding asset pricing formula with *physical* probabilities would be used to price assets. Similarly, outer risk neutral probabilities allow to price assets "as if" investors were risk averse and downside risk averse, but were neutral with respect to higher degree risks, including outer risk. The concepts of downside risk and outer risk neutral probabilities are thus natural extensions of the concept of risk neutral probabilities. In the paper, we provide interpretations of the changes in probability measures in terms of risk substitution.

We now explain the usefulness of downside risk neutral probabilities. First, downside risk neutral probabilities allow to value assets in a simple mean-variance framework. Even though mean-variance analysis (with physical probabilities) has at least since Rothschild and Stiglitz (1970) been criticized on the grounds that it does not take into account higher-order risk preferences, its simplicity and intuitive appeal are such that it remains a cornerstone of finance. The formulas that we present allow to apply mean-variance analysis in asset pricing, even though by construction they yield the same asset prices as the standard stochastic discount factor formula or the risk neutral pricing formula.

Second, the pricing kernel associated with downside risk neutral probabilities is linear in future wealth. This sets our analysis apart from a number of recent papers, such as Eraker (2008) and Martin (2013), which also derive new analytical expressions for asset prices, but in which the pricing kernel is not linear in state variables. These papers also assume CRRA or Epstein-Zin preferences, whereas we make minimal assumptions on the utility function. Linearity in state variables is advantageous for analytical tractability, interpretation, and empirical implementation (e.g., Chapman (1997), Brandt and Chapman (2013)). This tractability is achieved because the asset pricing formulas that involve downside risk neutral probabilities do not directly rely on the (nonlinear) utility function, but instead incorporate the coefficient of absolute risk aversion evaluated at the initial level of wealth. This aspect is reminiscent of the Arrow-Pratt approximation of the risk premium, which is widely used due to its simplicity and

[3] Other papers have already used a Taylor expansion of marginal utility for asset pricing purposes. Harvey and Siddique (2000), Dittmar (2002), and Chabi-Yo (2012), among others, approximate the pricing kernel with Taylor expansions of marginal utility of order two (or three), thus taking into account the effect of prudence (and temperance). Prudence and temperance have been defined as preferences over lotteries in Eeckhoudt and Schlesinger (2006).

its intuitive appeal – although the formulas that we derive in this paper are not approximations: they yield the same asset prices as other asset pricing formulas. A linear pricing kernel also allows to use the Capital Asset Pricing Model (CAPM) to price assets. The CAPM is widely used, but it requires strong assumptions. By contrast, with downside risk neutral probabilities, the CAPM holds with minimal assumptions.

We now explain the usefulness of outer risk neutral probabilities. First, the outer risk neutral probability measure can be viewed as a reasonable approximation of the physical probability measure. Indeed, there is strong direct and indirect empirical evidence for prudence (see Deck and Schlesinger (2010, 2014) and the references therein), which suggests that it is important to consider the aversion to downside risk in asset pricing. Direct evidence regarding temperance is more recent and more mixed (Deck and Schlesinger (2010, 2014)). In addition, whereas skewness has been identified as an important and robust determinant of asset returns, the effect of kurtosis has been found to be more limited, and does not show up in all specifications (Dittmar (2002), Chang, Christoffersen, and Jacobs (2013), Amaya, Christoffersen, Jacobs, Vasquez (2015)). Interestingly, outer risk neutral probabilities coincide with physical probabilities if and only if the utility function has zero temperance (whether or not it is prudent), a case which is not inconsistent with empirical findings (Deck and Schlesinger (2010)). This suggests that the outer risk neutral probability measure could be a close approximation of the physical probability measure, and could be used to improve tractability in asset pricing models at little cost.

Second, the pricing kernel associated with outer risk neutral probabilities is quadratic in future wealth. This can contribute to explain why empirical studies find that the pricing kernel is U-shaped. If the utility function is assumed to be risk averse at all levels of wealth, then a quadratic pricing kernel can only explain the first leg of this U-shape. If however temperance is indeed nil and prudence positive, then the (cubic) utility function becomes risk loving for sufficiently high levels of wealth,[4] which makes the pricing kernel nonmonotonic and U-shaped under the physical probability measure (if the utility function has zero temperance, then outer risk neutral probabilities coincide with physical probabilities). In this case, the risk neutral probability distribution has fat tails relative to the physical distribution, consistent with the empirical evidence. These results obtain in the microfounded standard asset pricing model. They contribute to a large literature which has proposed alternative theories and models to rationalize the U-shaped pricing kernel (e.g., Bakshi, Madan, and Panayotov (2010), Chabi-Yo (2012), Christoffersen, Heston, and Jabobs (2013)).

[4]The implication that the utility function is locally risk loving for sufficiently high levels of wealth is consistent with the evidence that some agents are locally risk loving (Deck and Schlesinger (2014), Noussair, Trautmann, and van de Kuilen (2014)). Besides, the actual risk preferences that matter for investment decisions, which have asset pricing implications, do not only reflect the agents' intrinsic preferences but also other factors such as taxation (see Tobin (1958), Mossin (1968), and Feldstein (1969)). Landier and Plantin (2011) argue that the fixed cost associated with tax avoidance may make intrinsically risk averse investors risk loving for sufficiently high levels of wealth.

Third, the model shows that explicitly taking into account downside risk aversion, as is possible with outer risk neutral probabilities, can lead to a reversal of common wisdom about the effect of volatility on asset prices (e.g., French, Schwert, and Stambaugh (1987)). Under the outer risk neutral probability measure, the price of assets with positive expected payoff, positive beta, and positive co-skewness is increasing in asset volatility and in the volatility of economic growth if and only if the coefficient of absolute prudence is sufficiently high, or equivalently if and only if wealth is sufficiently low under standard risk aversion. Indeed, an increase in volatility increases both the asset's beta and the asset's coskewness. This increases the asset's contribution to the risk of the wealth portfolio, but it also decreases its contribution to the downside risk of this portfolio. The latter effect dominates if and only if the level of wealth is sufficiently low. Thus, under natural assumptions on the utility function, the level of wealth in the economy plays an important role for asset prices comparative statics.

1 The model

For simplicity, we consider an economy with two dates: $t = 0$ and $t = 1$.[5] There is a representative agent, an expected utility maximizer with subjective discount factor $\beta \in (0, 1]$, and utility function u such that $u' > 0$. In addition, $u'' < 0$ if the agent is risk averse, $u''' > 0$ if the agent is prudent, and $u^{(4)} < 0$ if the agent is temperant (Scott and Horvath (1980)). Note that commonly used utility functions such as CARA and CRRA satisfy these assumptions. Current aggregate wealth in the economy is w_0, and future aggregate wealth is $\tilde{w}_1$. It is equal to w_s in state of the world s, for $s \in \{1, \ldots, S\}$, which occurs with probability $p_s \geq 0$, with $\sum_{s=1}^{S} p_s = 1$.

An asset is defined by its state-contingent $t = 1$ payoff $\tilde{x}$, which is equal to x_s in state of the world s. The values of w_0, w_s and x_s are finite for any s. We denote the riskfree rate in the economy by r_f. Using the standard stochastic discount factor formula (e.g. Hansen and Jagannathan (1991), Cochrane (2001, p.8)), the price at $t = 0$ of any given asset with payoff $\tilde{x}$ is

$$P = \mathbb{E}\left[\beta \frac{u'(\tilde{w}_1)}{u'(w_0)} \tilde{x}\right], \tag{1}$$

where $\beta \frac{u'(\tilde{w}_1)}{u'(w_0)}$ is the stochastic discount factor or pricing kernel, and $\mathbb{E}[\cdot]$ is the expectation operator with respect to the physical probability measure.

2 Risk neutral probabilities

Assume that u is of class C^2. For any given s, let $\eta_{2,s}$ be defined implicitly as

$$u'(w_s) \equiv \eta_{2,s} u'(w_0). \tag{2}$$

[5]It is straightforward but notationally cumbersome to extend the analysis to more than two dates.

Definition 1 *Let* $\nu_2 \equiv \sum_{s=1}^{S} p_s \eta_{2,s}$*, and*

$$\lambda_{2,s} \equiv \frac{p_s \eta_{2,s}}{\nu_2} = \frac{p_s \eta_{2,s}}{\sum_{s=1}^{S} p_s \eta_{2,s}}. \tag{3}$$

The set $\{\lambda_{2,s}\}$ *is the set of risk neutral probabilities, and* Λ_2 *is the risk neutral probability measure.*

By construction, $\sum_{s=1}^{S} \lambda_{2,s} = 1$. Note that $\frac{d\lambda_{2,s}}{dp_s} = \frac{\eta_{2,s}}{\nu_2}$ is the Radon-Nikodym derivative of the risk neutral measure with respect to the physical measure. With linear utility, the risk neutral probability measure coincides with the physical probability measure: $u'(w_s) = u'(w_0)$ for any w_s, so that $\eta_{2,s} = \frac{\eta_{2,s}}{\nu_2} = 1$ for any s.

We now briefly study the determinants of the divergence between the physical and the risk neutral probability measure, i.e., we study the determinants of $\eta_{2,s}$. First, we have $\frac{d\eta_{2,s}}{dw_s} \leq 0$ if $u'' \leq 0$, with a strict inequality if $u'' < 0$.[6] Intuitively, with respect to the physical probability measure, the risk neutral probability measure overweighs bad states of the world, and underweighs good states of the world. This is illustrated in Figure 1, which depicts the determinants of $\eta_{2,s}$, namely $u'(w_s)$ and $u'(w_0)$ (cf. equation (2)) as a function of w_s. Second, if $w_s > w_0$ (respectively $w_s < w_0$), then according to the mean value theorem (cf. Simon and Blume (1994), p.825) there exists $y_s \in (w_0, w_s)$ (resp. $y_s \in (w_s, w_0)$) such that:

$$u'(w_s) = u'(w_0) + u''(y_s)(w_s - w_0) \tag{4}$$

Therefore, with $u' > 0$ and $u'' < 0$, we have $\eta_{2,s} > 1$ if and only if $w_s < w_0$. Moreover, given that $u'(w_s) > 0$ and $u'(w_0) > 0$, (2) implies that $\eta_{2,s} > 0$ for any s. Combining (2) and (4) gives

$$\eta_{2,s} = \frac{u'(w_0) + u''(y_s)(w_s - w_0)}{u'(w_0)} = 1 + \frac{u''(y_s)(w_s - w_0)}{u'(w_0)} \tag{5}$$

This equation shows that $\eta_{2,s}$ depends on the degree of concavity of the utility function. For a utility function with constant u'', equation (5) simply rewrites as $\eta_{2,s} = 1 - A(w_0)(w_s - w_0)$, where $A(w_0) \equiv -\frac{u''(w_0)}{u'(w_0)}$ is the coefficient of absolute risk aversion at w_0.

The price of any asset can be expressed with risk neutral probabilities. Substituting $u'(w_s)$ from (2) in (1), the price P of an asset with stochastic payoff $\tilde{x}$ may be rewritten as:

$$P = \sum_{s=1}^{S} p_s \left[\beta \frac{\eta_{2,s} u'(w_0)}{u'(w_0)} x_s\right] = \beta \nu_2 \mathbb{E}^{\Lambda_2}[\tilde{x}], \tag{6}$$

where $\mathbb{E}^{\Lambda_2}[\cdot]$ is the expectation operator with respect to the probability measure Λ_2. Given that this formula must hold for any asset, including the riskfree asset whose payoff is $x_s = 1$ for all s and whose price is by definition of the riskfree rate r_f equal to $P = \frac{1}{1+r_f}$, we have $\frac{1}{1+r_f} = \beta\nu_2$ (the

[6]The proof immediately follows from (2) and the fact that u' is decreasing with $u'' < 0$, and constant with $u'' = 0$.

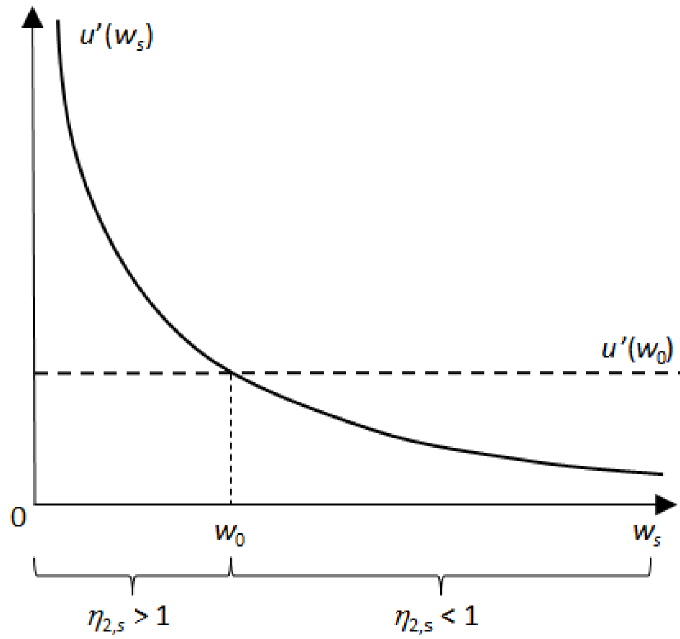

Figure 1: The derivation of risk neutral probabilities with $u'' < 0$ (cf. equations (2) and (3)).

expectation of a constant under any probability measure is equal to this constant). Substituting in (6) gives

$$P = \frac{\mathbb{E}^{\Lambda_2}\left[\tilde{x}\right]}{1 + r_f} \tag{7}$$

which is the standard asset pricing formula with risk neutral probabilities.

3 Downside risk neutral probabilities

Assume that u is of class C^3. For any given s, let $\eta_{3,s}$ be defined implicitly as ($\eta_{3,s}$ exists generically – except for $u'(w_0) + u''(w_0)(w_s - w_0) = 0$):

$$u'(w_s) \equiv \eta_{3,s}\left[u'(w_0) + u''(w_0)(w_s - w_0)\right]. \tag{8}$$

We henceforth consider economies such that $\eta_{3,s}$ exists for all s.

Definition 2 *Let* $\nu_3 \equiv \sum_{s=1}^{S} p_s \eta_{3,s}$, *and*

$$\lambda_{3,s} \equiv \frac{p_s \eta_{3,s}}{\nu_3} = \frac{p_s \eta_{3,s}}{\sum_{s=1}^{S} p_s \eta_{3,s}}. \tag{9}$$

The set $\{\lambda_{3,s}\}$ *is the set of downside risk neutral probabilities, and* Λ_3 *is the downside risk neutral probability measure.*

By construction, $\sum_{s=1}^{S} \lambda_{3,s} = 1$. Note that $\frac{\eta_{3,s}}{\nu_3}$ is the Radon-Nikodym derivative of the downside risk neutral measure with respect to the physical measure. With linear utility or quadratic utility ($u''' = 0$ in both cases), the downside risk neutral probability measure coincides with the physical probability measure: by construction, the Taylor expansion $u'(w_0) + u''(w_0)(w_s - w_0)$ is then equal to $u'(w_s)$ for any w_s, so that $\eta_{3,s} = \frac{\eta_{3,s}}{\nu_3} = 1$ for any s.

We now study the determinants of the divergence between the physical and the downside risk neutral probability measure, as measured by $\eta_{3,s}$, when $u''' > 0$. The evidence suggests that absolute risk aversion $A(c)$ is nonincreasing in wealth (e.g., Levy (1994), Chiappori and Paiella (2011)), i.e., it is either constant (CARA) or decreasing (DARA) (note that CRRA utility is DARA).[7] Under this assumption, we have the following relation between $\eta_{3,s}$ and future wealth:

Claim 1 *Suppose that $u'' < 0$. If the utility function is CARA or DARA and if $\eta_{3,s}$ exists, then*

$$\frac{d\eta_{3,s}}{dw_s} \gtreqqless 0 \text{ if } w_s \gtreqqless w_0.$$

Proof. Rewrite (8) as

$$\eta_{3,s} = \frac{u'(w_s)}{u'(w_0) + u''(w_0)(w_s - w_0)}, \tag{10}$$

so that

$$\frac{d\eta_{3,s}}{dw_s} = \frac{u''(w_s)\left(u'(w_0) + u''(w_0)(w_s - w_0)\right) - u'(w_s)u''(w_0)}{\left(u'(w_0) + u''(w_0)(w_s - w_0)\right)^2}. \tag{11}$$

The denominator in (11) is positive, and the numerator can be rearranged as

$$\underbrace{u'(w_0)u''(w_s) - u'(w_s)u''(w_0)}_{\text{A}} + \underbrace{u''(w_0)u''(w_s)(w_s - w_0)}_{\text{B}} \tag{12}$$

The sign of A is

$$\text{sign(A)} = \text{sign}\left(u'(w_0)u''(w_s) - u'(w_s)u''(w_0)\right) \tag{13}$$

$$= \text{sign}\left(\frac{u'(w_0)u''(w_s) - u'(w_s)u''(w_0)}{u'(w_0)u'(w_s)}\right) \tag{14}$$

$$= \text{sign}\left(\frac{u''(w_s)}{u'(w_s)} - \frac{u''(w_0)}{u'(w_0)}\right) \tag{15}$$

If the utility function is CARA, then sign(A) $= 0$. If the utility function is DARA, then sign(A) $\gtreqqless 0$ if $w_s - w_0 \gtreqqless 0$. For $u'' < 0$ (whether the utility function is CARA or DARA), we have sign(B) $\gtreqqless 0$ if $w_s - w_0 \gtreqqless 0$. ■

To better understand the determinants of the divergence between the downside risk neutral probability and the physical probability, as measured by $\eta_{3,s}$, we now study the difference

[7] In the standard version of the portfolio choice problem with a risky asset and a riskfree asset, the dollar amount invested in the risky asset is increasing in wealth if and only if the utility function is DARA (e.g. Gollier (2001) p.59).

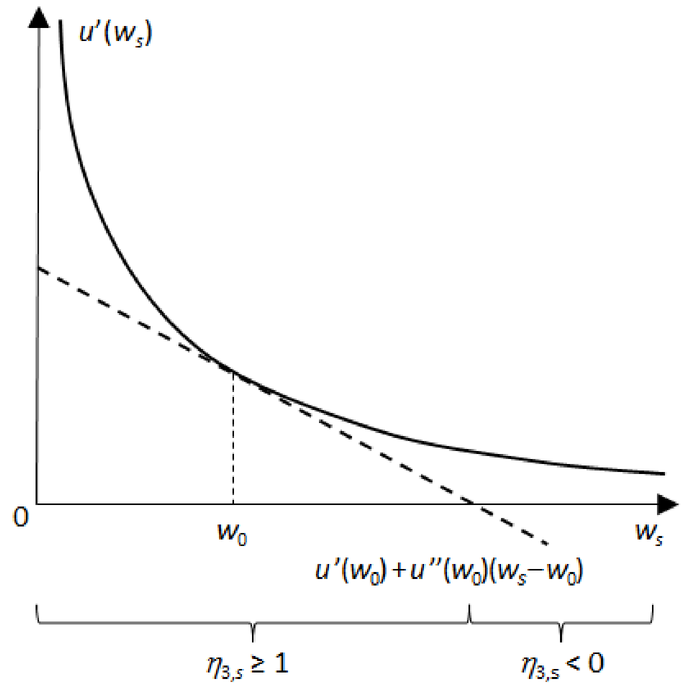

Figure 2: The derivation of downside risk neutral probabilities with $u'' < 0$ and $u''' > 0$ (cf. equations (8) and (9)).

between $u'(w_s)$ and the term in brackets on the right-hand-side of (8). If $w_s > w_0$ (respectively $w_s < w_0$), then according to Theorem 30.5 in Simon and Blume (1994, p.828) there exists $z_s \in (w_0, w_s)$ (resp. $z_s \in (w_s, w_0)$) such that:

$$u'(w_s) = u'(w_0) + u''(w_0)\,(w_s - w_0) + \frac{1}{2}u'''(z_s)\,(w_s - w_0)^2 \tag{16}$$

Therefore, with $u' > 0$, $u'' < 0$ and $u''' > 0$, we have $\eta_{3,s} > 1$ if $w_s < w_0$. This is because $u''' > 0$ means that u' is convex, so that u' lies above its tangents. However, we do not necessarily have $\eta_{3,s} > 1$ if $w_s > w_0$, because the term in brackets on the right-hand-side of (8) can then be negative, in which case $\eta_{3,s} < 0$.[8] This is because a first-order Taylor expansion of marginal utility is negative when w_s is high enough (in the same way that marginal utility is negative for a high enough argument of the utility function with quadratic utility), so that $\eta_{3,s}$ must also be negative for (8) to hold. This is illustrated in Figure 2, which depicts the determinants of $\eta_{3,s}$,

[8] According to Dirac (1942), "Negative energies and probabilities should not be considered as nonsense. They are well-defined concepts mathematically, like a negative sum of money, since the equations which express the important properties of energies and probabilities can still be used when they are negative." Like risk neutral probabilities, downside risk neutral probabilities are a mathematical construct. They are not "physical" probabilities, i.e., they do not represent the probability of occurrence of some events. Instead, their purpose is to provide alternative pricing operators – the fact that some of them can be negative is not inherently problematic in that regard.

namely $u'(w_s)$ and $u'(w_0) + u''(w_0)(w_s - w_0)$ (cf. equation (8)) as a function of w_s. Combining (8) and (16) gives

$$\eta_{3,s} = \frac{u'(w_0) + u''(w_0)\,(w_s - w_0) + \frac{1}{2}u'''(z_s)\,(w_s - w_0)^2}{u'(w_0) + u''(w_0)(w_s - w_0)} = 1 + \frac{1}{2}\frac{u'''(z_s)\,(w_s - w_0)^2}{u'(w_0) + u''(w_0)(w_s - w_0)} \quad (17)$$

Intuitively, when the agent has an aversion to both downside risk and higher degrees risk, so that the utility function is neither linear nor quadratic, $\eta_{3,s}$ is adjusted to take into account these higher-order risk preferences. In particular, for an agent who is averse to risk, to downside risk, but not to outer risk, i.e. with $u'' < 0$, $u''' > 0$, and $u^{(4)} = 0$ (which implies $u'''(z_s) = u'''(w_0)$ for any z_s, and $u'' < 0$ given the assumption $u'' \leq 0$), equation (17) rewrites as

$$\eta_{3,s} = 1 + \frac{1}{2}\left(\frac{1}{D(w_0)}\frac{1}{(w_s - w_0)^2} - \frac{1}{P(w_0)}\frac{1}{w_s - w_0}\right)^{-1}, \quad (18)$$

where $D(w_0) \equiv \frac{u'''(w_0)}{u'(w_0)}$ is the coefficient of downside risk aversion at w_0 (Modica and Scarsini (2005), Crainich and Eeckhoudt (2008), Keenan and Snow (2010)), and $P(w_0) \equiv -\frac{u'''(w_0)}{u''(w_0)}$ is the coefficient of absolute prudence at w_0 (Kimball (1990)). Kraus and Litzenberger (1976) postulate $u''' > 0$ and $u^{(4)} = 0$ to derive a "three moments CAPM", while Harvey and Siddique (2000) use a second-order Taylor expansion of marginal utility, which includes u''' but omits $u^{(4)}$.

The economic intuition for the expression in (18) is the following. An investor with quadratic utility only has preferences for the first and second degrees risk of the distribution. A first degree risk deterioration is a change in the distribution which is undesirable in the sense of first-order stochastic dominance, i.e., for all agents with increasing utility functions. A second degree risk increase is a change in the distribution which leaves the mean unchanged but is undesirable for all agents with concave utility functions (Rothschild and Stiglitz (1970)). Note that a first degree risk improvement implies a higher mean, and an increase in second degree risk implies a higher variance. The change in probability measure described in $\eta_{3,s}$ alters the first and second degrees risk of the distribution of states of the world to incorporate the effect of higher degree risks. In particular, with $u^{(4)} = 0$, the change in probability measure alters the first and second degrees risk to incorporate the effect of third degree risk or "downside risk". This change therefore depends on the relation between the aversion to downside risk, and the aversion to the first and second degrees risk of the distribution. This explains why $D(w_0)$ and $P(w_0)$ appear in (18): Liu and Meyer (2013) show that $(-1)^{m-1}\frac{u^{(3)}(w_0)}{u^{(m)}(w_0)}$ for $m = 1, 2$, is a measure of the rate of substitution between an mth degree risk increase and an increase in third degree risk, i.e., a measure of the willingness to increase an mth degree risk to avoid an increase in downside risk.

We now show how to express the price of any asset with downside risk neutral probabilities. Substituting $u'(w_s)$ from (8) in (1) gives

$$P = \sum_{s=1}^{S} p_s \left[\beta\eta_{3,s} \left(1 + \frac{u''(w_0)}{u'(w_0)}(w_s - w_0) \right) x_s \right] \tag{19}$$

$$= \beta\nu_3 \mathbb{E}^{\Lambda_3} \left[\tilde{x} + \frac{u''(w_0)}{u'(w_0)}(\tilde{w}_1 - w_0)\tilde{x} \right]$$

$$= \beta\nu_3 \left(\mathbb{E}^{\Lambda_3}[\tilde{x}] - A(w_0)\mathbb{E}^{\Lambda_3}[(\tilde{w}_1 - w_0)\tilde{x}] \right) \tag{20}$$

Note that the asset price P in these formulas and in Propositions 1 and 2 below is the same as the asset price in equations (1) and (7). With u of class C^3, the price P of an asset with stochastic payoff $\tilde{x}$ may be decomposed in the following terms:

Proposition 1

$$P = \frac{1}{1+r_f} \mathbb{E}^{\Lambda_3} \left[\frac{f(w_0, \tilde{w}_1)}{\mathbb{E}^{\Lambda_3}[f(w_0, \tilde{w}_1)]} \tilde{x} \right] \tag{21}$$

$$= \frac{1}{1+r_f} \left[\mathbb{E}^{\Lambda_3}[\tilde{x}] - \frac{A(w_0) cov^{\Lambda_3}(\tilde{w}_1, \tilde{x})}{1 - A(w_0)\mathbb{E}^{\Lambda_3}[\tilde{w}_1 - w_0]} \right] \tag{22}$$

where $f(w_0, \tilde{w}_1)$ *is linear in* $\tilde{w}_1$*, and writes as* $f(w_0, \tilde{w}_1) \equiv 1 - A(w_0)[\tilde{w}_1 - w_0]$.

Proof. Given that equation (20) must hold for any asset, including the riskfree asset whose payoff is $x_s = 1$ for all s and whose price is by definition of the riskfree rate r_f equal to $P = \frac{1}{1+r_f}$, we have

$$\frac{1}{1+r_f} = \beta\nu_3 \left(1 - A(w_0)\mathbb{E}^{\Lambda_3}[\tilde{w}_1 - w_0] \right).$$

Substituting in (20) gives

$$P = \frac{1}{1+r_f} \frac{\mathbb{E}^{\Lambda_3}[\tilde{x}] - A(w_0)\mathbb{E}^{\Lambda_3}[(\tilde{w}_1 - w_0)\tilde{x}]}{1 - A(w_0)\mathbb{E}^{\Lambda_3}[\tilde{w}_1 - w_0]}. \tag{23}$$

This formula can be rewritten as in (21) or (22). ∎

For preferences such that $u''' = 0$, and maintaining the assumption that $u'(w_s) > 0$ for any s (with $u'' < 0$, this implies that w_s is bounded from above), Λ_3 coincides with the physical probability measure, and the utility function can without loss of generality be written as $u(w) = w - \frac{b}{2}w^2$, with $b \geq 0$ if $u'' \leq 0$. The price of any asset with stochastic payoff $\tilde{x}$ is then[9]

$$P = \frac{1}{1+r_f} \left[\mathbb{E}[\tilde{x}] - \frac{A(w_0) cov(\tilde{w}_1, \tilde{x})}{1 - A(w_0)\mathbb{E}[\tilde{w}_1 - w_0]} \right] = \frac{1}{1+r_f} \left[\mathbb{E}[\tilde{x}] - \frac{b\, cov(\tilde{w}_1, \tilde{x})}{1 - b\,\mathbb{E}[\tilde{w}_1]} \right]. \tag{24}$$

In formula (22), the change in probability measure also takes into account the asset price impact of preferences for downside risk and higher degree risks. It is important to note that, in (22), the

[9]As above, the assumption that $u'(w_s) > 0$ for any s guarantees that $1 - A(w_0)\mathbb{E}[\tilde{w}_1 - w_0]$ or equivalently $1 - b\mathbb{E}[\tilde{w}_1]$ is strictly positive.

expression $1 - A(w_0)\mathbb{E}^{\Lambda_3}[\tilde{w}_1 - w_0]$ is strictly positive, as shown in the Supplementary Appendix. We have $\mathbb{E}^{\Lambda_3}[\tilde{w}_1 - w_0] = 0$ if the expected growth in wealth under Λ_3 is nil; $\mathbb{E}^{\Lambda_3}[\tilde{x}] = 0$ if the expected asset payoff under Λ_3 is nil; and $cov^{\Lambda_3}(\tilde{w}_1, \tilde{x}) = 0$ if the asset payoff is uncorrelated with aggregate wealth under Λ_3.

In equation (21), the term $\frac{1}{1+r_f}\frac{f(w_0,\tilde{w}_1)}{\mathbb{E}^{\Lambda_3}[f(w_0,\tilde{w}_1)]}$ can be viewed as the pricing kernel associated with downside risk neutral probabilities. Comparing (22) and (24) shows that this pricing kernel corresponds to the one that would obtain with quadratic utility. That is, $\frac{1}{1+r_f}\frac{f(w_0,\tilde{w}_1)}{\mathbb{E}[f(w_0,\tilde{w}_1)]}$ corresponds to the pricing kernel in a world where agents only have preferences about the mean and the variance of the distribution of their future wealth (with quadratic utility, the expected utility associated with any probability distribution is fully described by its mean and its variance). Thus, using downside risk neutral probabilities allows to price assets in a mean-variance framework. Moreover, equation (21) and the definition of $f(w_0, \tilde{w}_1)$ show that the pricing kernel associated with downside risk neutral probabilities is linear in $\tilde{w}_1$, in contrast with the stochastic discount factor formula in (1).

As is well-known, a linear pricing kernel allows to use the CAPM to derive the expected return on a security. Denoting by $\tilde{R}_i \equiv \frac{\tilde{x}}{P}$ the gross return on a given security i with payoff $\tilde{x}$, by $\tilde{R}_w$ the gross return on the wealth portfolio with payoff $\tilde{w}_1$, and by $\beta_i^{\Lambda_3} \equiv \frac{cov^{\Lambda_3}(\tilde{R}_i,\tilde{R}_w)}{var^{\Lambda_3}(\tilde{R}_w)}$ the security's CAPM beta under the downside risk neutral measure, we have:

$$\mathbb{E}^{\Lambda_3}[\tilde{R}_i] - R_f = \beta_i^{\Lambda_3}\left[\mathbb{E}^{\Lambda_3}[\tilde{R}_w] - R_f\right] \tag{25}$$

We refer to the Supplementary appendix for technical details. Crucially, whereas the CAPM with physical probabilities requires strong assumptions, the CAPM that can be derived with downside risk neutral probabilities requires minimal assumptions.

Comparing the expression in (22) with the one in (7) shows that using downside risk neutral probabilities (the probability measure Λ_3) instead of risk neutral probabilities (the probability measure Λ_2) results in the apparition of an additional term in the asset pricing formula. Indeed, while the risk neutral probability measure is adjusted to take into account aversion to second and higher degree risks, the downside risk neutral probability measure is only adjusted to take into account aversion to third and higher degree risks. Comparing the expression in (22) with the one in (24) shows that, with downside risk neutral probabilities, assets can be valued "as if" agents were only averse to first and second degree risks but were neutral with respect to higher degree risks (including downside risk). Higher-order risk preferences such as aversion to downside risk and to outer risk are incorporated in asset prices via a change in the probability measure.

The second term on the right-hand-side of (22) can thus be interpreted as the correction for second degree risk. Aversion to second degree risk, i.e., risk aversion, is captured by $u'' < 0$, and it implies $A(w_0) > 0$. Equation (22) then shows that an asset whose payoff $\tilde{x}$ is positively correlated with future wealth $\tilde{w}_1$ has a lower price. In equation (22), the utility function only (directly) enters the equation via the coefficient of *absolute* risk aversion evaluated at the initial level of wealth. This formula also enables us to study the effect of risk aversion on asset prices independently of the effect of higher-order risk preferences.

Indeed, by holding the downside risk neutral probability measure constant, the model enables us to study the effect of changes in different parameters on the asset price via their effect on the fundamentals that matter to an investor averse to the first two degrees of risk only, holding the effect of higher-order risk preferences constant. The following Corollary, which is proven in the Supplementary Appendix, provides comparative statics of a given asset price with respect to its volatility, expected economic growth, and economic growth volatility under the downside risk neutral measure. Note that changes in economic growth volatility leave expected economic growth constant, and changes in asset volatility leave the expected asset payoff constant.

Corollary 1 *Suppose that* $u'' < 0$*, let* $\beta^{\Lambda_3}(\tilde{x}, \tilde{w}_1) \equiv \frac{cov^{\Lambda_3}(\tilde{x},\tilde{w}_1)}{var^{\Lambda_3}(\tilde{w}_1)}$*,* $\tilde{w}_1 \equiv w_0 + a + \sigma\tilde{\omega}$*, where* $\mathbb{E}^{\Lambda_3}[\tilde{\omega}] = 0$*,* σ *is strictly positive and finite,* $var^{\Lambda_3}(\tilde{\omega}) = 1$*, and* $\tilde{x} \equiv k + \sigma_x\tilde{\varepsilon}$*, where* $\mathbb{E}^{\Lambda_3}[\tilde{\varepsilon}] = 0$ *and* σ_x *is strictly positive and finite.*[10] *For given probability measure* Λ_3*,*

$$\frac{\partial}{\partial a}P < 0 \qquad \Leftrightarrow \qquad \mathbb{E}^{\Lambda_3}[\tilde{x}] > 0 \tag{26}$$

$$\frac{\partial}{\partial \sigma}P < 0 \qquad \Leftrightarrow \qquad \beta^{\Lambda_3}(\tilde{x}, \tilde{w}_1) > 0 \tag{27}$$

$$\frac{\partial}{\partial \sigma_x}P < 0 \qquad \Leftrightarrow \qquad \beta^{\Lambda_3}(\tilde{x}, \tilde{w}_1) > 0 \tag{28}$$

The comparative statics in Corollary (1) are intuitive. In equation (26), an asset with a positive expected payoff under the downside risk neutral measure is less valuable if aggregate wealth growth (or "economic growth") is higher due to an intertemporal consumption smoothing effect. In equations (27) and (28), an increase in either asset volatility or economic volatility leads to a lower asset price if the asset contributes positively to the volatility of the portfolio (i.e., it has a positive beta) under the downside risk neutral measure, because this type of asset increases investors' risk exposure. While these comparative statics under the downside risk neutral measure may seem straightforward, note that they do not necessarily hold under the outer risk neutral measure (or under the physical measure), as will be shown in Corollary 2 in the next section.

In Corollary 4 in the Supplementary Appendix, we conduct comparative statics of the second term in brackets on the right-hand-side of (22) with respect to initial wealth w_0 and to expected "economic growth", for a given Λ_3. Holding the downside risk neutral measure constant allows to study the change in asset price due to preferences for the first two degrees of risk only, holding

[10] Note that the decompositions of $\tilde{w}_1$ and $\tilde{x}$ without loss of generality, since a random variable can be decomposed into its expectation and an additive noise term, and a constant can be decomposed into a given constant and an additional term. The only restrictions imposed are that we consider strictly positive macroeconomic uncertainty ($\sigma > 0$), and risky assets ($\sigma_x > 0$). It is straightforward but notationally cumbersome to extend the analysis to the cases of $\sigma = 0$ and/or $\sigma_x = 0$. Moreover, we show in the Supplementary Appendix that, for a given probability measure Λ, the CAPM beta of an asset, $\beta_{\tilde{x}}$, writes as $\beta_{\tilde{x}} = \frac{w_0}{P}\beta^{\Lambda}(\tilde{x}, \tilde{w}_1)$.

constant the asset price effect of higher-order risk preferences. With decreasing absolute risk aversion, we show that the adjustment for risk in the sense of mean-preserving spreads will be especially important relative to the asset's expected payoff in a fast-growing economy with a low initial level of wealth, but that it will matter less in a more wealthy, slow-growing economy.

4 Outer risk neutral probabilities

Assume that u is of class C^4. For any given s, let $\eta_{4,s}$ be defined implicitly as

$$u'(w_s) \equiv \eta_{4,s}\left[u'(w_0) + u''(w_0)(w_s - w_0) + \frac{1}{2}u'''(w_0)(w_s - w_0)^2\right]. \tag{29}$$

Definition 3 *Let* $\nu_4 \equiv \sum_{s=1}^{S} p_s \eta_{4,s}$, *and*

$$\lambda_{4,s} \equiv p_s \frac{\eta_{4,s}}{\nu_4} = \frac{p_s \eta_{4,s}}{\sum_{s=1}^{S} p_s \eta_{4,s}}. \tag{30}$$

The set $\{\lambda_{4,s}\}$ *is the set of outer risk neutral probabilities, and* Λ_4 *is the outer risk neutral probability measure.*

By construction, $\sum_{s=1}^{S} \lambda_{4,s} = 1$. Note that $\frac{\eta_{4,s}}{\nu_4}$ is the Radon-Nikodym derivative of the outer risk neutral measure with respect to the physical measure. With linear, quadratic or cubic utility ($u^{(4)} = 0$ in all three cases), we show below that the outer risk neutral probability measure coincides with the physical probability measure. More generally, with $u^{(4)} \leq 0$, if $w_s > w_0$ (respectively $w_s < w_0$), then according to Theorem 30.6 in Simon and Blume (1994, p.829) there exists $\zeta_s \in (w_0, w_s)$ (resp. $\zeta_s \in (w_s, w_0)$) such that:

$$u'(w_s) = u'(w_0) + u''(w_0)\,(w_s - w_0) + \frac{1}{2}u'''(w_0)\,(w_s - w_0)^2 + \frac{1}{6}u^{(4)}(\zeta_s)\,(w_s - w_0)^3 \tag{31}$$

Comparing this equality with the one in (29) yields the following result, which is proved in the Supplementary Appendix:

Claim 2 *With* $u^{(4)} = 0$, $\eta_{4,s}$ *exists and is equal to 1 for any* s. *With* $u^{(4)} < 0$ *and* $u'' \leq 0$, $\eta_{4,s}$ *exists and is strictly positive for any* s, *and* $\eta_{4,s} \lesseqqgtr 1$ *for* $w_s \gtreqqless w_0$.

The second part of Claim 2 is illustrated in Figure 3, which depicts the determinants of $\eta_{4,s}$, namely $u'(w_s)$ and $u'(w_0) + u''(w_0)(w_s - w_0) + \frac{1}{2}u'''(w_0)(w_s - w_0)^2$ (cf. equation (29)) as a function of w_s. To further study the determinants of $\eta_{4,s}$, combine (29) and (31) to get

$$\begin{aligned} \eta_{4,s} &= \frac{u'(w_0) + u''(w_0)\,(w_s - w_0) + \frac{1}{2}u'''(w_0)\,(w_s - w_0)^2 + \frac{1}{6}u^{(4)}(\zeta_s)\,(w_s - w_0)^3}{u'(w_0) + u''(w_0)(w_s - w_0) + \frac{1}{2}u'''(w_0)\,(w_s - w_0)^2} \\ &= 1 + \frac{1}{6}\frac{u^{(4)}(\zeta_s)\,(w_s - w_0)^3}{u'(w_0) + u''(w_0)(w_s - w_0) + \frac{1}{2}u'''(w_0)\,(w_s - w_0)^2} \end{aligned} \tag{32}$$

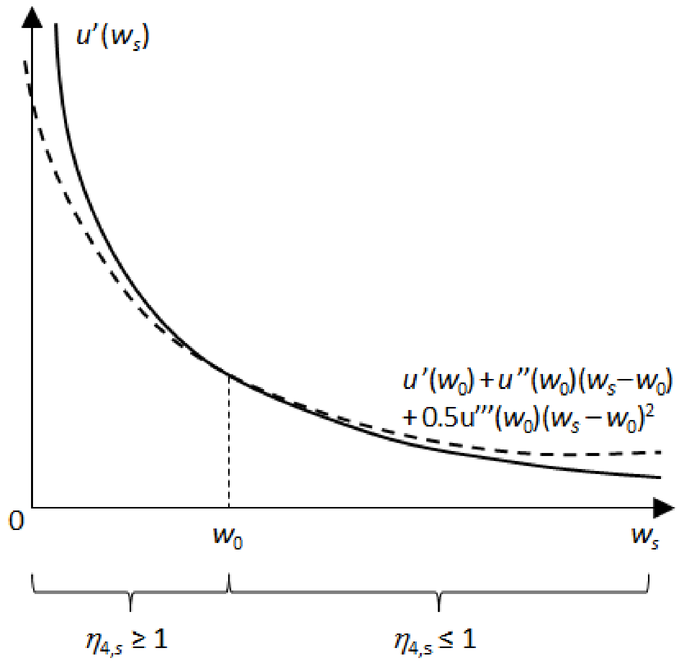

Figure 3: The derivation of outer risk neutral probabilities with $u'' < 0$, $u''' > 0$, and $u^{(4)} < 0$ (cf. equations (29) and (30))

In particular, for an agent who does not have a preference for higher degree risks than fourth degree risk or outer risk, i.e. with $u^{(5)} = 0$, equation (32) rewrites as

$$\eta_{4,s} = 1 + \frac{1}{6}\left(\frac{1}{\frac{u^{(4)}(w_0)}{u'(w_0)}(w_s - w_0)^3} + \frac{1}{\frac{u^{(4)}(w_0)}{u''(w_0)}(w_s - w_0)^2} + \frac{1}{2\frac{u^{(4)}(w_0)}{u'''(w_0)}(w_s - w_0)}\right)^{-1}, \tag{33}$$

where $-(-1)^{m-1}\frac{u^{(4)}(w_0)}{u^{(m)}(w_0)}$, for $m = 1, 2, 3$, are all measures of the intensity of temperance (Crainich and Eeckhoudt (2011), Liu and Meyer (2013)). The approximation of the pricing kernel in Dittmar (2002, equation (6)) considers preferences for the first four degrees of risk only, which is consistent with $u^{(5)} = 0$.

The economic intuition for the expression in (33) is that the aforementioned first three degrees of risk – including downside risk – of the probability distribution of states of the world are altered to incorporate the effect of higher degree risks. In particular, with $u^{(5)} = 0$, the change in probability measure alters the first three degrees of risk to incorporate the effect of fourth degree risk or "outer risk". This change therefore depends on the relation between the aversion to outer risk, and the aversion to the first three degrees of risk. This explains why several measures of the intensity of aversion to outer risk or "temperance" appear in (33): Liu and Meyer (2013) show that $-(-1)^{m-1}\frac{u^{(4)}(w_0)}{u^{(m)}(w_0)}$ for $m = 1, 2, 3$, is a measure of the rate of substitution between an mth degree risk increase and an increase in fourth degree risk, i.e., a measure of the willingness to increase an mth degree risk to avoid an increase in outer risk.

We now show how to express the price of any asset with outer risk neutral probabilities. Substituting $u'(w_s)$ from (29) in (1), the asset price P may be rewritten as:

$$\begin{aligned} P &= \sum_{s=1}^{S} p_s \left[\beta \eta_{4,s} \left(1 + \frac{u''(w_0)}{u'(w_0)}(w_s - w_0) + \frac{1}{2}\frac{u'''(w_0)}{u'(w_0)}(w_s - w_0)^2 \right) x_s \right] \\ &= \beta \nu_4 \mathbb{E}^{\Lambda_4} \left[\tilde{x} - A(w_0)(\tilde{w}_1 - w_0)\tilde{x} + \frac{1}{2} D(w_0)(\tilde{w}_1 - w_0)^2 \tilde{x} \right] \\ &= \beta \nu_4 \left(\mathbb{E}^{\Lambda_4}[\tilde{x}] - A(w_0)\mathbb{E}^{\Lambda_4}\left[(\tilde{w}_1 - w_0)\tilde{x}\right] + \frac{1}{2} D(w_0)\mathbb{E}^{\Lambda_4}\left[(\tilde{w}_1 - w_0)^2 \tilde{x}\right] \right) \end{aligned} \quad (34)$$

where $A(w_0)$ is the coefficient of absolute risk aversion at w_0, and $D(w_0) \equiv \frac{u'''(w_0)}{u'(w_0)}$ is the coefficient of downside risk aversion at w_0. With u of class C^4, the price P of an asset with stochastic payoff $\tilde{x}$ may be decomposed in the following terms:

Proposition 2

$$P = \frac{1}{1+r_f} \mathbb{E}^{\Lambda_4} \left[\frac{g(w_0, \tilde{w}_1)}{\mathbb{E}^{\Lambda_4}[g(w_0, \tilde{w}_1)]} \tilde{x} \right] \quad (35)$$

$$= \frac{1}{1+r_f} \left[\mathbb{E}^{\Lambda_4}[\tilde{x}] + \frac{-A(w_0) cov^{\Lambda_4}(\tilde{w}_1, \tilde{x}) + \frac{1}{2} D(w_0) cov^{\Lambda_4}((\tilde{w}_1 - w_0)^2, \tilde{x})}{1 - A(w_0)\mathbb{E}^{\Lambda_4}[\tilde{w}_1 - w_0] + \frac{1}{2} D(w_0)\mathbb{E}^{\Lambda_4}[(\tilde{w}_1 - w_0)^2]} \right] \quad (36)$$

where $g(w_0, \tilde{w}_1) \equiv 1 - A(w_0)[\tilde{w}_1 - w_0] + \frac{1}{2} D(w_0)(\tilde{w}_1 - w_0)^2$.

Proof. Given that equation (34) must hold for any asset, including the riskfree asset whose payoff is $x_s = 1$ for all s and whose price is by definition of the riskfree rate r_f equal to $P = \frac{1}{1+r_f}$, we have

$$\frac{1}{1+r_f} = \beta \nu_4 \left(1 - A(w_0)\mathbb{E}^{\Lambda_4}[\tilde{w}_1 - w_0] + \frac{1}{2} D(w_0)\mathbb{E}^{\Lambda_4}[(\tilde{w}_1 - w_0)^2] \right). \quad (37)$$

Substituting (37) in (34) gives

$$P = \frac{1}{1+r_f} \frac{\mathbb{E}^{\Lambda_4}[\tilde{x}] - A(w_0)\mathbb{E}^{\Lambda_4}[(\tilde{w}_1 - w_0)\tilde{x}] + \frac{1}{2} D(w_0)\mathbb{E}^{\Lambda_4}[(\tilde{w}_1 - w_0)^2 \tilde{x}]}{1 - A(w_0)\mathbb{E}^{\Lambda_4}[\tilde{w}_1 - w_0] + \frac{1}{2} D(w_0)\mathbb{E}^{\Lambda_4}[(\tilde{w}_1 - w_0)^2]}. \quad (38)$$

This formula can be rewritten as in (35) or (36). ■

The intuition behind equation (36) is that an asset whose payoff $\tilde{x}$ positively covaries with future wealth $\tilde{w}_1$ will have a lower price if the agent is risk averse ($A(w_0) > 0$), as in the previous section. In addition, an asset whose payoff tends to be low when future wealth deviates more from its current level will have a lower price if the agent is averse to downside risk ($D(w_0) > 0$). Consistent with the relation between downside risk and skewness, it is noteworthy that, up to a scaling factor, the term $cov\left((\tilde{w}_1 - w_0)^2, \tilde{x}\right)$ can be interpreted as the co-skewness of the asset (Harvey and Siddique (2000), Chabi-Yo, Leisen, and Renault (2014)). In formula (36), the opposite of this covariance measures the contribution of the asset to the downside risk of the wealth portfolio under the outer risk neutral measure (that is, a negative covariance means a positive contribution to downside risk). It is also important to note that, in (36), the expression $1-A(w_0)\mathbb{E}^{\Lambda_4}[\tilde{w}_1-w_0]+\frac{1}{2}D(w_0)\mathbb{E}^{\Lambda_4}[(\tilde{w}_1-w_0)^2]$ is strictly positive, as shown in the Supplementary Appendix.

As in the previous section, in equation (35), the term $\frac{1}{1+r_f}\frac{g(w_0,\tilde{w}_1)}{\mathbb{E}^{\Lambda_4}[g(w_0,\tilde{w}_1)]}$ can be viewed as the pricing kernel associated with outer risk neutral probabilities. It is quadratic in future wealth according to Proposition 2. If $u'' < 0$ at all levels of wealth, then the pricing kernel is decreasing in wealth.

We also show in the Supplementary Appendix that the expected return on any asset i can be expressed as

$$\mathbb{E}^{\Lambda_4}[\tilde{R}_i] - R_f = \chi\, cov^{\Lambda_4}(\tilde{R}_w, \tilde{R}_i) + \vartheta\, cov^{\Lambda_4}(\tilde{R}_w^2, \tilde{R}_i), \tag{39}$$

for two constants χ and ϑ. This result is similar to equation (7) in Harvey and Siddique (2000), but there is an important difference. Harvey and Siddique (2000) use physical probabilities and assume a quadratic stochastic discount factor, so that their result is only an approximation if the stochastic discount factor is not quadratic in wealth. By contrast, using outer risk neutral probabilities ensures that the equation in (39) holds in any case. With $u^{(4)} = 0$, outer risk neutral probabilities coincide with physical probabilities, and our result coincides with Harvey and Siddique's. With $u^{(4)} \neq 0$, outer risk neutral probabilities will diverge from physical probabilities in such a way that (39) still holds.

For preferences such that the utility function has zero temperance (i.e., $u^{(4)} = 0$), the outer risk neutral probability measure Λ_4 coincides with the physical probability measure (cf. Claim 2). The price of any asset with stochastic payoff $\tilde{x}$ is then

$$P = \frac{1}{1+r_f}\left[\mathbb{E}[\tilde{x}] + \frac{-A(w_0)cov\,(\tilde{w}_1,\tilde{x}) + \frac{1}{2}D(w_0)cov\,((\tilde{w}_1-w_0)^2,\tilde{x})}{1-A(w_0)\mathbb{E}[\tilde{w}_1-w_0]+\frac{1}{2}D(w_0)\mathbb{E}[(\tilde{w}_1-w_0)^2]}\right]. \tag{40}$$

The difference between this formula and the one in (36) is that, in (36), the change in probability measure also takes into account the asset price impact of preferences for outer risk and higher degree risks.

For $u^{(4)} = 0$, we now show that the pricing kernel associated with the outer risk neutral probability measure in Proposition 2 is equal to the stochastic discount factor in (1), is quadratic in wealth, and is increasing in wealth for sufficiently high levels of wealth:

Claim 3 *With $u''' > 0$ and $u^{(4)} = 0$, the pricing kernel $\frac{1}{1+r_f}\frac{g(w_0,\tilde{w}_1)}{\mathbb{E}[g(w_0,\tilde{w}_1)]}$ is equal to $\beta\frac{u'(\tilde{w}_1)}{u'(w_0)}$, it is quadratic in future wealth w_s, and $\frac{d}{dw_s}\frac{u'(w_s)}{u'(w_0)} \lesseqqgtr 0$ for $w_s \lesseqqgtr \bar{w}$. In addition, $u''(w_s) \lesseqqgtr 0$ if and only if $w_s \lesseqqgtr \bar{w}$.*

Proof. With $u^{(4)} = 0$, we know from Claim 2 that Λ_4 coincides with the physical probability measure. Then, because the expressions in (1) and (35) must hold for any distribution of asset payoffs and are both equal to P, we have $\frac{1}{1+r_f}\frac{g(w_0,w_s)}{\mathbb{E}[g(w_0,w_s)]} = \beta\frac{u'(w_s)}{u'(w_0)}$ for all w_s. With

$u^{(4)} = 0$, for any w_s we can write without loss of generality $u(w_s) = w_s - \frac{b}{2}w_s^2 + \frac{c}{3}w_s^3$, so that $u'(w_s) = 1 - bw_s + cw_s^2$ (where $c > 0$ due to $u''' > 0$). Then $\beta\frac{u'(w_s)}{u'(w_0)} = \beta\frac{1-bw_s+cw_s^2}{1-bw_0+cw_0^2}$, which is quadratic in w_s. Furthermore, $\frac{d}{dw_s}\frac{u'(w_s)}{u'(w_0)} = \frac{-b+2cw_s}{u'(w_0)}$, where $u'(w_0) > 0$ and $-b + 2cw_s \lesseqqgtr 0$ if and only if $w_s \lesseqqgtr \frac{b}{2c} \equiv \bar{w}$. Finally, $u''(w_s) = -b+2cw_s$, so that $u''(w_s) \lesseqqgtr 0$ if and only if $w_s \lesseqqgtr \frac{b}{2c} \equiv \bar{w}$. ■

When the utility function is prudent but has zero temperance, the pricing kernel is non-monotone: it is quadratic in wealth, and it becomes increasing in wealth for sufficiently high levels of wealth ($w_s > \bar{w}$), which are such that the utility function is locally risk loving. The existing empirical evidence is not inconsistent with these risk preferences. Indeed, the first paper that directly tests for aversion to outer risk or temperance, Deck and Schlesinger (2010), does not find support for temperance. In subsequent analyzes, Deck and Schlesinger (2014) and Noussair, Trautmann, and van de Kuilen (2014) find that although a majority of the population is (locally) temperate and risk averse, a substantial fraction of the population is (locally) intemperate and risk loving (note that risk lovers can be prudent, as shown by Crainich, Eeckhoudt, and Trannoy (2013)). Deck and Schlesinger (2014) conclude: "Although risk lovers might be in a minority, it is perhaps surprising that more attention has not been given to their potential behavior." Noussair, Trautmann, and van de Kuilen (2014) also find that temperance is associated with less risky investment portfolios and with higher risk aversion, suggesting that less temperate investors may be more active in financial markets.

In addition, when the utility function is prudent but has zero temperance, the risk neutral probability distribution has fat tails relative to the physical distribution. Indeed, with $u^{(4)} = 0$:

$$\eta_{2,s} \equiv \frac{u'(w_s)}{u'(w_0)} = \frac{u'(w_0) + u''(w_0)(w_s - w_0) + \frac{1}{2}u'''(w_0)(w_s - w_0)^2}{u'(w_0)} \tag{41}$$

$$= 1 + A(w_0)(w_s - w_0) + \frac{D(w_0)}{2}(w_s - w_0)^2. \tag{42}$$

With $u''' > 0$ and therefore $D(w_0) > 0$, this implies that $\eta_{2,s}$ is especially large for very high and very low values of w_s. Given that $\eta_{2,s}$ is the (scaled) ratio of the risk neutral to the physical probability in state s ($\eta_{2,s} \equiv \frac{\lambda_{2,s}}{p_s}\nu_2$), this in turn implies that the risk neutral probability distribution has fat tails relative to the physical distribution. Equation (42) also emphasizes the importance of downside risk aversion in the Radon-Nikodym derivative of the risk neutral measure with respect to the physical measure, $\frac{\eta_{2,s}}{\nu_2}$. Downside risk aversion is especially important in explaining the divergence between the risk neutral and the physical probability for levels of future wealth w_s that differ substantially from the current level of wealth w_0. More precisely, the effect of changes in downside risk aversion on $\eta_{2,s}$ dominates the effect of changes in risk aversion, in the sense that $\left|\frac{d\eta_{2,s}}{dD(w_0)}\right| > \left|\frac{d\eta_{2,s}}{dA(w_0)}\right|$, for $|w_s - w_0| > 2$. In Appendix C, we show how the model with $u^{(4)} = 0$ can also shed light on the empirical finding that the risk neutral volatility of an asset tends to be higher than its physical volatility. We notably show that the coefficient of absolute risk aversion and the coefficient of downside risk aversion play important roles in explaining the difference between the risk neutral and the physical volatility.

By holding the outer risk neutral probability measure constant, the model enables us to study the effect of changes in different parameters on the asset price via their effect on the fundamentals that matter to an investor averse to the first three degrees of risk only, holding the effect of higher-order risk preferences constant. The following Corollary of Proposition 2, which is proven in the Supplementary Appendix, provides comparative statics of a given asset price with respect to its volatility, expected economic growth, and economic growth volatility under the outer risk neutral measure. As above, note that changes in economic growth volatility leave expected economic growth constant, and changes in asset volatility leave the expected asset payoff constant.

Corollary 2 *Suppose that $u'' < 0$, $u''' > 0$, let $\beta^{\Lambda_4}(\tilde{x}, \tilde{w}_1) \equiv \frac{cov^{\Lambda_4}(\tilde{x},\tilde{w}_1)}{var^{\Lambda_4}(\tilde{w}_1)}$, $\tilde{w}_1 \equiv w_0 + a + \sigma\tilde{\omega}$, where $\mathbb{E}^{\Lambda_4}[\tilde{\omega}] = 0$, $var^{\Lambda_4}(\tilde{\omega}) = 1$, σ is strictly positive and finite, and $\tilde{x} \equiv k + \sigma_x \tilde{\varepsilon}$, where $\mathbb{E}^{\Lambda_4}[\tilde{\varepsilon}] = 0$, $var^{\Lambda_4}[\tilde{\varepsilon}] = 1$, and σ_x is strictly positive and finite.[11] For given probability measure Λ_4,*

$$\text{For } \mathbb{E}^{\Lambda_4}[\tilde{x}] > 0, \ \frac{\partial}{\partial a}P > 0 \quad \Leftrightarrow \quad \frac{1}{P(w_0)} < a + \sigma^2 \frac{\beta^{\Lambda_4}(\tilde{x}, \tilde{w}_1)}{\mathbb{E}^{\Lambda_4}[\tilde{x}]} \tag{43}$$

$$\text{For } \mathbb{E}^{\Lambda_4}[\tilde{x}] < 0, \ \frac{\partial}{\partial a}P > 0 \quad \Leftrightarrow \quad \frac{1}{P(w_0)} > a + \sigma^2 \frac{\beta^{\Lambda_4}(\tilde{x}, \tilde{w}_1)}{\mathbb{E}^{\Lambda_4}[\tilde{x}]} \tag{44}$$

$$\text{For } \beta^{\Lambda_4}(\tilde{x}, \tilde{w}_1) > 0, \ \frac{\partial}{\partial \sigma}P > 0 \quad \Leftrightarrow \quad \frac{1}{P(w_0)} < a + \frac{E^{\Lambda_4}[\tilde{x}]}{\beta^{\Lambda_4}(\tilde{x}, \tilde{w}_1)} + \frac{cov^{\Lambda_4}(\tilde{\omega}^2, \tilde{x})}{\beta^{\Lambda_4}(\tilde{x}, \tilde{w}_1)} \tag{45}$$

$$\text{For } \beta^{\Lambda_4}(\tilde{x}, \tilde{w}_1) < 0, \ \frac{\partial}{\partial \sigma}P > 0 \quad \Leftrightarrow \quad \frac{1}{P(w_0)} > a + \frac{E^{\Lambda_4}[\tilde{x}]}{\beta^{\Lambda_4}(\tilde{x}, \tilde{w}_1)} + \frac{cov^{\Lambda_4}(\tilde{\omega}^2, \tilde{x})}{\beta^{\Lambda_4}(\tilde{x}, \tilde{w}_1)} \tag{46}$$

$$\text{For } \beta^{\Lambda_4}(\tilde{x}, \tilde{w}_1) > 0, \ \frac{\partial}{\partial \sigma_x}P > 0 \quad \Leftrightarrow \quad \frac{1}{P(w_0)} < a + \frac{\sigma}{2} \frac{cov^{\Lambda_4}(\tilde{\omega}^2, \tilde{x})}{cov^{\Lambda_4}(\tilde{\omega}, \tilde{x})} \tag{47}$$

$$\text{For } \beta^{\Lambda_4}(\tilde{x}, \tilde{w}_1) < 0, \ \frac{\partial}{\partial \sigma_x}P > 0 \quad \Leftrightarrow \quad \frac{1}{P(w_0)} > a + \frac{\sigma}{2} \frac{cov^{\Lambda_4}(\tilde{\omega}^2, \tilde{x})}{cov^{\Lambda_4}(\tilde{\omega}, \tilde{x})} \tag{48}$$

Seemingly intuitive comparative statics in Corollary 1 can be reversed if downside risk aversion is sufficiently important – remember that in Corollary 1, Λ_3 was held constant, i.e., *changes* in the asset price due to downside risk aversion and higher order risk preferences were omitted. First, as shown in (43) and (44), the price of an asset can be positively related to economic growth, especially if the asset beta is positive. In this case, higher economic growth further decreases the asset's contribution to the downside risk of the wealth portfolio under the outer risk neutral measure. Second, as shown in (45)-(48), an increase in either asset volatility or economic volatility which leaves the asset expected payoff constant can lead to a higher (respectively lower) asset price, especially if the term $cov^{\Lambda_4}(\tilde{\omega}^2, \tilde{x})$ is positive (resp. negative) – the opposite of this term measures the asset's contribution to the downside risk of the wealth portfolio under the outer risk neutral measure. Indeed, higher volatility increases the absolute

[11] As mentioned above, these decompositions are without loss of generality as long as $\tilde{w}_1$ and $\tilde{x}$ are stochastic. Moreover, we show in the Supplementary Appendix that, for a given probability measure Λ, the CAPM beta of an asset, $\beta_{\tilde{x}}$, writes as $\beta_{\tilde{x}} = \frac{w_0}{P}\beta^{\Lambda}(\tilde{x}, \tilde{w}_1)$.

value of this term, and further decreases (resp. increases) the asset's contribution to the downside risk of the wealth portfolio under the outer risk neutral measure. An increase in economic volatility σ also triggers a precautionary saving effect (which is apparent in the second terms on the right-hand-sides of (45) and (46)), which tends to increase the asset price if the asset's expected payoff under the outer risk neutral measure is positive.

The model thus emphasizes that changes in macroeconomic volatility or in the asset's volatility, which have implications for the asset's co-skewness, can lead to a reversal of common theoretical wisdom. For example, for an asset with a positive beta and a positive co-skewness, an increase in asset volatility leads not only to an increase in asset risk (as measured by its beta), but also to a decrease in asset downside risk (as measured by the opposite of its co-skewness). The latter effect can outweigh the former and lead to a higher asset price if downside risk aversion is sufficiently high relative to risk aversion. This is the case if the coefficient of absolute prudence $P(w_0)$ is sufficiently high.

Under standard risk aversion, the coefficient of absolute prudence $P(w_0)$ is decreasing in w_0. Then, a negative shock to wealth w_0 (think about an economic crisis or a market crash) increases $P(w_0)$, and is such that the conditions in (43), (45), (47) (respectively (44), (46), (48)) hold for a larger (resp. smaller) set of parameter values. Standard risk aversion was introduced by Kimball (1993) (it has already been used in the asset pricing literature, e.g., Weil (1992), Dittmar (2002), Gollier and Schlesinger (2002), and Chang, Christoffersen, and Jacobs (2013)). It implies that the existence of an uninsurable background risk reduces the optimal investment in a risky asset with an independent return. In sum, under the outer risk neutral measure, the comparative statics of asset prices with respect to changes in economic conditions and in asset volatility depend on the relative importance of downside risk aversion and risk aversion, which in turn depends on the level of wealth in the economy under standard risk aversion. Changes in this level of wealth may thus trigger "regime shifts", which alter the reaction of asset prices to changes in asset volatilities and to changes in the expectation and the volatility of economic growth under the outer risk neutral measure.

The coefficient of absolute prudence, already defined below equation (18), can be rewritten as $P(w_0) = \frac{D(w_0)}{A(w_0)}$. In this sense, the coefficient of absolute prudence measures the importance of downside risk aversion relative to risk aversion in conditions (43)-(48). This observation contributes to the debate on the appropriate measure of downside risk aversion (e.g., Chiu (2005), Crainich and Eeckhoudt (2008)), and it suggests the coefficient of absolute prudence as a measure of the tradeoff between risk aversion and downside risk aversion in asset pricing. With CRRA utility, $P(w_0) = \frac{1+\gamma}{w_0}$.[12] More generally, under the assumption of standard risk aversion (Kimball (1993)), the coefficient of absolute prudence, $P(w_0)$, is decreasing in w_0.

The macroeconomic environment, described by a and σ, also plays an important role in determining whether the downside risk aversion effect or the risk aversion effect dominates in

[12]With CRRA utility, an increase in relative risk aversion γ, which increases $P(w_0)$, leads (perhaps misleadingly) to an increase in the importance of downside risk aversion relative to risk aversion.

Corollary 2 (recall that only the risk aversion effect is present in Corollary 1). For example, as $a \to 0$ and $\sigma \to 0$, conditions (43) and (47) do not hold, in which case the risk aversion effect dominates. Studying the relative importance of the terms on the numerator of the second term in brackets in (36) further shows that, in an economy with low economic growth and low economic volatility, the effect of downside risk aversion is negligible compared to the effect of risk aversion under the outer risk neutral measure:

Corollary 3 *Using the same notations as in Corollary 2, for given probability measure* Λ_4 *and with* $cov^{\Lambda_4}(\tilde{\omega}, \tilde{\varepsilon}) \neq 0$,

$$\lim_{a \to 0, \sigma \to 0} \frac{cov^{\Lambda_4}\left((\tilde{w}_1 - w_0)^2, \tilde{x}\right)}{cov^{\Lambda_4}\left(\tilde{w}_1, \tilde{x}\right)} = 0 \tag{49}$$

This result suggests that, with low economic growth and low economic volatility, the adjustment for third degree risk (as captured by the term $\frac{1}{2}D(w_0)cov^{\Lambda_4}((\tilde{w}_1 - w_0)^2, \tilde{x})$ in (36), up to a scaling factor) will only have a minor impact on asset prices relative to the adjustment for second degree risk (as captured by the term $-A(w_0)cov^{\Lambda_4}(\tilde{w}_1, \tilde{x})$ in (36), up to the same scaling factor).

The results and discussions in this section have highlighted that either omitting or mis-specifying risk preferences of higher order than risk aversion can lead to erroneous conclusions. They have also established that the asset price impact of risk preferences depends on the macroeconomic environment and on aggregate wealth, so that this impact could vary over time. It would be interesting to determine whether mis-specifications of higher order risk preferences can explain apparent failures of standard asset pricing models, or whether the success of "alternative" modeling devices can be in part explained by their ability to somehow better take into account the effect of higher order risk preferences on asset prices.

5 Conclusion

This paper introduces the concepts of downside (respectively outer) risk neutral probabilities, which are adjusted to take into account the aversion to downside (resp. outer) risk and higher degree risks. While risk neutral probabilities allow to value assets in a risk neutral framework, downside risk neutral probabilities allow to value assets in a simple and intuitive mean-variance framework. With downside (resp. outer) risk neutral probabilities, the pricing kernel is linear (resp. quadratic) in future wealth. The changes in probability measure described in this paper can thus improve tractability in asset pricing models. The new asset pricing formula that we derive also further our understanding of the effect of risk aversion, downside risk aversion, and prudence on asset prices. In Appendix A, we extend the analysis by introducing i-th degree risk neutral probabilities. In Appendix B, we plot the densities that correspond to different probability measures in a special case for illustrative purposes.

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A Generalization

This Appendix generalizes the analysis by introducing i-th degree risk neutral probabilities and the associated asset pricing formulas.

Assume that u is of class C^i, for an integer $i \geq 2$. For given i and s, let $\eta_{i,s}$ be defined implicitly as ($\eta_{i,s}$ exists generically – except for $\sum_{k=0}^{i-2} \frac{1}{k!} u^{(k+1)}(w_0)(w_s - w_0)^k = 0$):

$$u'(w_s) \equiv \eta_{i,s} \sum_{k=0}^{i-2} \frac{1}{k!} u^{(k+1)}(w_0)(w_s - w_0)^k. \tag{50}$$

Definition 4 *Let* $\nu_i \equiv \sum_{s=1}^{S} p_s \eta_{i,s}$, *and*

$$\lambda_{i,s} \equiv p_s \frac{\eta_{i,s}}{\nu_i} = \frac{p_s \eta_{i,s}}{\sum_{s=1}^{S} p_s \eta_{i,s}}. \tag{51}$$

The set $\{\lambda_{i,s}\}$ *is the set of* i*-th degree risk neutral probabilities, and* Λ_i *is the* i*-th degree risk neutral probability measure.*

We now show how to express the price of any asset with i-th degree risk neutral probabilities. Substituting $u'(w_s)$ from (50) in (1), the asset price P may be decomposed in the following terms:

$$\begin{aligned} P &= \sum_{s=1}^{S} p_s \left[\beta \eta_{i,s} \sum_{k=0}^{i-2} \frac{1}{k!} \frac{u^{(k+1)}(w_0)}{u'(w_0)} (w_s - w_0)^k \tilde{x} \right] \\ &= \beta \nu_i \mathbb{E}^{\Lambda_i} \left[\sum_{k=0}^{i-2} \frac{1}{k!} C_{k+1}(w_0)(\tilde{w}_1 - w_0)^k \tilde{x} \right] \\ &= \beta \nu_i \left[\sum_{k=0}^{i-2} \frac{1}{k!} C_{k+1}(w_0) \mathbb{E}^{\Lambda_i} \left[(\tilde{w}_1 - w_0)^k \tilde{x} \right] \right] \end{aligned} \tag{52}$$

where $C_{k+1}(w_0) \equiv \frac{u^{(k+1)}(w_0)}{u'(w_0)}$ is the coefficient of absolute preference for the $k+1$-th degree risk at w_0. A negative preference (i.e., an aversion) for the $k+1$-th degree risk implies that $C_{k+1}(w_0) < 0$. The price P of an asset with stochastic payoff $\tilde{x}$ may be decomposed in the following terms:

Proposition 3 *If* u *is of class* C^i*, for a given integer* $i \geq 2$*:*

$$P = \frac{1}{1+r_f} \frac{\sum_{k=0}^{i-2} \frac{1}{k!} C_{k+1}(w_0) \mathbb{E}^{\Lambda_i} \left[(\tilde{w}_1 - w_0)^k \tilde{x} \right]}{\sum_{k=0}^{i-2} \frac{1}{k!} C_{k+1}(w_0) \mathbb{E}^{\Lambda_i} \left[(\tilde{w}_1 - w_0)^k \right]} \tag{53}$$

$$= \frac{1}{1+r_f} \mathbb{E}^{\Lambda_i} \left[\frac{f_i(w_0, \tilde{w}_1)}{\mathbb{E}^{\Lambda_i}[f_i(w_0, \tilde{w}_1)]} \tilde{x} \right] \tag{54}$$

where $f_i(w_0, \tilde{w}_1) \equiv \sum_{k=0}^{i-2} \frac{1}{k!} C_{k+1}(w_0)(\tilde{w}_1 - w_0)^k$.

Proof. Given that equation (52) must hold for any asset, including the riskfree asset whose payoff is $x_s = 1$ for all s and whose price is by definition of the riskfree rate r_f equal to $P = \frac{1}{1+r_f}$, we have

$$\frac{1}{1+r_f} = \beta \nu_i \left[\sum_{k=0}^{i-2} \frac{1}{k!} C_{k+1}(w_0) \mathbb{E}^{\Lambda_i} \left[(\tilde{w}_1 - w_0)^k \right] \right]. \tag{55}$$

Substituting (55) in (52) gives (53). ■

In particular, for preferences such that $u^{(i)} = 0$, Λ_i coincides with the physical probability measure, and the price of any asset with stochastic payoff $\tilde{x}$ is

$$P = \frac{1}{1+r_f} \frac{\sum_{k=0}^{i-2} \frac{1}{k!} C_{k+1}(w_0) \mathbb{E}\left[(\tilde{w}_1 - w_0)^k \tilde{x} \right]}{\sum_{k=0}^{i-2} \frac{1}{k!} C_{k+1}(w_0) \mathbb{E}\left[(\tilde{w}_1 - w_0)^k \right]}. \tag{56}$$

Finally, we study the limit case as $i \to \infty$. Assume that u' is analytic around w_0. For any given s, let $\eta_{\infty,s}$ be defined implicitly as ($\eta_{\infty,s}$ exists generically – except for $\sum_{k=0}^{\infty} \frac{1}{k!} u^{(k+1)}(w_0)(w_s - w_0)^k = 0$):

$$u'(w_s) \equiv \eta_{\infty,s} \sum_{k=0}^{\infty} \frac{1}{k!} u^{(k+1)}(w_0)(w_s - w_0)^k. \tag{57}$$

Proposition 4 *If u' is analytic around w_0, the i-th degree risk neutral probability measure, $\Lambda_{\infty,s}$, coincides with the physical probability measure when $i \to \infty$.*

Proof. Given that u' is analytic around w_0, the Taylor series of u' give:

$$u'(w_s) = \sum_{k=0}^{\infty} \frac{1}{k!} u^{(k+1)}(w_0)(w_s - w_0)^k. \tag{58}$$

Comparing with (57), this implies that $\eta_{\infty,s} = 1$ for any s. Then Definition 4 implies that $\lambda_{\infty,s} = p_s$ for any s. ■

B Numerical example

In this section, we set $w_0 = 1$, we let $\tilde{w}_1$ be normally distributed with a mean of zero and a variance of one, and we consider a CARA utility function with an absolute risk aversion of $\frac{1}{2}$.[13] We use the analog of the formulas presented in the paper for continuous distributions.

Figures 4 to 7 depict successively the physical density, the risk neutral density, the downside risk neutral density, and the outer risk neutral density of states of the world. Note that the risk neutral density is the one that differs the most from the physical density at first glance. This is expected because the risk neutral distribution is adjusted for second and higher degree risks,

[13] This specification was chosen in part to avoid problems that occurred with other probability distributions or utility functions when computing ν_i, for $i \in \{2, 3, 4\}$. Note that using a symmetric distribution facilitates comparisons between probability measures.

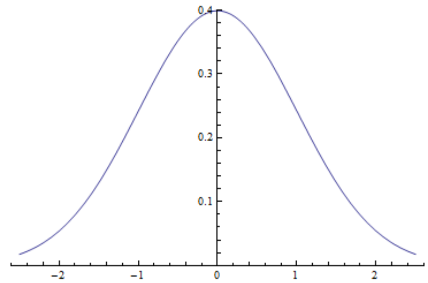

Figure 4: The physical density.

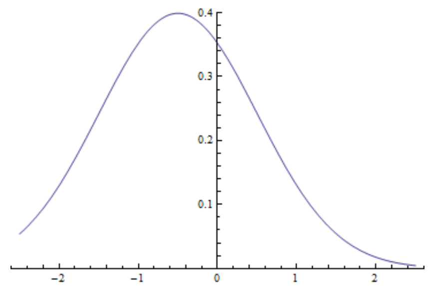

Figure 5: The risk neutral density.

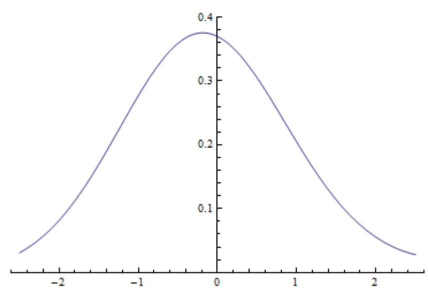

Figure 6: The downside risk neutral density.

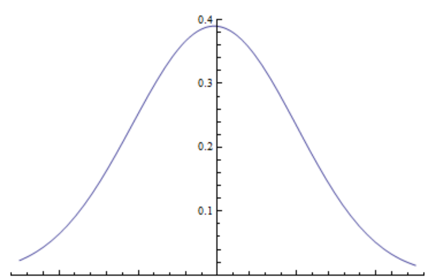

Figure 7: The outer risk neutral density.

whereas the downside risk neutral distribution is only adjusted for third and higher degree risks, and the outer risk neutral distribution is only adjusted for fourth and higher degree risks. Note that the outer risk neutral density is very similar to the physical density, but it is still different from the physical density because CARA utility is characterized by $u^{(4)} \neq 0$.

C Risk neutral volatility and physical volatility

This Appendix compares the risk neutral volatility and the physical volatility. We rewrite the variance of the asset payoff under the physical measure, then under the risk neutral measure:

$$var(\tilde{x}) = \mathbb{E}\left[\tilde{x}^2\right] - \left(\mathbb{E}\left[\tilde{x}\right]\right)^2 = \sum_s p_s x_s^2 - \left(\sum_s p_s x_s\right)^2 . \tag{59}$$

$$\begin{aligned} var^{\Lambda_2}(\tilde{x}) &= \mathbb{E}^{\Lambda_2}\left[\tilde{x}^2\right] - \left(\mathbb{E}^{\Lambda_2}\left[\tilde{x}\right]\right)^2 = \sum_s \lambda_{2,s} x_s^2 - \left(\sum_s \lambda_{2,s} x_s\right)^2 \\ &= \sum_s p_s \frac{\eta_{2,s}}{\nu_2} x_s^2 - \left(\sum_s p_s \frac{\eta_{2,s}}{\nu_2} x_s\right)^2 = \mathbb{E}\left[\frac{\tilde{\eta}_2}{\nu_2}\tilde{x}^2\right] - \left(\mathbb{E}\left[\frac{\tilde{\eta}_2}{\nu_2}\tilde{x}\right]\right)^2 \\ &= \mathbb{E}\left[\tilde{x}^2\right] + cov\left(\frac{\tilde{\eta}_2}{\nu_2}, \tilde{x}^2\right) - \left(\mathbb{E}\left[\tilde{x}\right] + cov\left(\frac{\tilde{\eta}_2}{\nu_2}, \tilde{x}\right)\right)^2 , \end{aligned} \tag{60}$$

where in the last equality we used $\mathbb{E}\left[\frac{\eta_{2,s}}{\nu_2}\right] = 1$, by definition of ν_2. Comparing (59) and (60) and removing offsetting terms, we have $var^{\Lambda_2}(\tilde{x}) > var(\tilde{x})$ if and only if:

$$cov\left(\frac{\tilde{\eta}_2}{\nu_2}, \tilde{x}^2\right) - 2\mathbb{E}[\tilde{x}]cov\left(\frac{\tilde{\eta}_2}{\nu_2}, \tilde{x}\right) - \left(cov\left(\frac{\tilde{\eta}_2}{\nu_2}, \tilde{x}\right)\right)^2 > 0 \tag{61}$$

This makes clear that the variance of the asset payoff $\tilde{x}$ is higher under the risk neutral measure than under the physical measure if and only if $cov\left(\frac{\tilde{\eta}_2}{\nu_2}, \tilde{x}^2\right)$ is sufficiently high, given $cov\left(\frac{\tilde{\eta}_2}{\nu_2}, \tilde{x}\right)$.

Moreover, with $u^{(4)} = 0$, and using (42), these scaled covariances can be rewritten as:

$$cov\left(\tilde{\eta}_2, \tilde{x}^2\right) = A(w_0)cov\left(\tilde{w}_1, \tilde{x}^2\right) + \frac{D(w_0)}{2}cov\left((\tilde{w}_1 - w_0)^2, \tilde{x}^2\right), \tag{62}$$

$$cov\left(\tilde{\eta}_2, \tilde{x}\right) = A(w_0)cov\left(\tilde{w}_1, \tilde{x}\right) + \frac{D(w_0)}{2}cov\left((\tilde{w}_1 - w_0)^2, \tilde{x}\right). \tag{63}$$

With the assumption that $u^{(4)} = 0$, these expressions make clear how covariances, coskewnesses, and risk preferences affect the difference between the asset variance under the risk neutral and the physical distribution, which appears on the left-hand-side of (61).

Supplementary Appendix for Downside risk neutral probabilities
Not for publication

The expression $1 - A(w_0)\mathbb{E}^{\Lambda_3}\left[\tilde{w}_1 - w_0\right]$ in (22)

We show that this expression is strictly positive.

First, consider the states s such that $u'(w_0) + u''(w_0)(w_s - w_0) > 0$, and therefore $\eta_{3,s} > 0$ (given (8) and $u' > 0$), which implies $\lambda_{3,s} \geq 0$. Then we have $\lambda_{3,s}\left[u'(w_0) + u''(w_0)(w_s - w_0)\right] \geq 0$. Dividing each term by $u'(w_0)$, which is strictly positive, and summing over these states s, we have

$$\sum_{s|\eta_{3,s}>0} \lambda_{3,s}\left[1 - A(w_0)(w_s - w_0)\right] \geq 0. \tag{64}$$

Second, consider the states s such that $u'(w_0) + u''(w_0)(w_s - w_0) < 0$, and therefore $\eta_{3,s} < 0$ (given (8) and $u' > 0$), which implies $\lambda_{3,s} \leq 0$. Then we have $\lambda_{3,s}\left[u'(w_0) + u''(w_0)(w_s - w_0)\right] \geq 0$. Dividing each term by $u'(w_0)$, which is strictly positive, and summing over these states s, we have

$$\sum_{s|\eta_{3,s}<0} \lambda_{3,s}\left[1 - A(w_0)(w_s - w_0)\right] \geq 0. \tag{65}$$

Third, because $\lambda_{3,s}\left[1 - A(w_0)(w_s - w_0)\right] \geq 0$ for any s and $1 - A(w_0)(w_s - w_0) \neq 0$ (otherwise $\eta_{3,s}$ would not exist), both expressions on the left-hand-sides of (64) and (65) will be equal to zero only if $\lambda_{3,s} = 0$ for all s, which would imply $\sum_s \lambda_{3,s} \neq 1$, a contradiction. Therefore, at least one of the expressions on the left-hand-sides of (64) and (65) is strictly positive, while the other is nonnegative. Adding up these two sums then yields

$$\sum_s \lambda_{3,s}\left[1 - A(w_0)(w_s - w_0)\right] = 1 - A(w_0)\mathbb{E}^{\Lambda_3}\left[\tilde{w}_1 - w_0\right] > 0.$$

Linear pricing kernel and the CAPM

Let $\tilde{m} \equiv \frac{1}{1+r_f}\frac{f(w_0,\tilde{w}_1)}{\mathbb{E}^{\Lambda_3}[f(w_0,\tilde{w}_1)]}$. The pricing kernel $\tilde{m}$ in Proposition 1 is linear in $\tilde{w}_1$, and can thus be expressed as:

$$\tilde{m} = a + b\tilde{R}_w, \tag{66}$$

where $\tilde{R}_w \equiv \frac{\tilde{w}_1}{P_w}$ is the gross return on the wealth portfolio with $t = 0$ price P_w, and

$$a \equiv \frac{1}{1+r_f}\frac{1 + A(w_0)w_0}{1 - A(w_0)(\mathbb{E}^{\Lambda_3}[\tilde{w}_1] - w_0)}, \qquad b \equiv \frac{1}{1+r_f}\frac{-A(w_0)P_w}{1 - A(w_0)(\mathbb{E}^{\Lambda_3}[\tilde{w}_1] - w_0)}. \tag{67}$$

Equation (21) can thus be rewritten as $P = \mathbb{E}^{\Lambda_3}[\tilde{m}\tilde{x}]$. Dividing both sides by P gives $1 = \mathbb{E}^{\Lambda_3}[\tilde{m}\tilde{R}_i]$, where $\tilde{R}_i \equiv \frac{\tilde{x}}{P}$ is the gross return of asset i. In turn, this equation can be rewritten as

$$\frac{1}{\mathbb{E}^{\Lambda_3}[\tilde{m}]} = \mathbb{E}^{\Lambda_3}[\tilde{R}_i] + \frac{cov^{\Lambda_3}(\tilde{m}, \tilde{R}_i)}{\mathbb{E}^{\Lambda_3}[\tilde{m}]}. \tag{68}$$

Using $\mathbb{E}^{\Lambda_3}[\tilde{m}] = \frac{1}{1+r_f} \equiv \frac{1}{R_f}$, and substituting for $\tilde{m}$, the equation above rewrites as

$$\mathbb{E}^{\Lambda_3}[\tilde{R}_i] - R_f = -\frac{b\, cov^{\Lambda_3}(\tilde{R}_w, \tilde{R}_i)}{\mathbb{E}^{\Lambda_3}[\tilde{m}]}. \tag{69}$$

In particular, if asset i is the wealth portfolio with gross return $\tilde{R}_w$, we have

$$\mathbb{E}^{\Lambda_3}[\tilde{R}_w] - R_f = -\frac{b\, var^{\Lambda_3}(\tilde{R}_w)}{\mathbb{E}^{\Lambda_3}[\tilde{m}]}. \tag{70}$$

Equating (69) and (70),

$$\mathbb{E}^{\Lambda_3}[\tilde{R}_i] - R_f = \frac{cov^{\Lambda_3}(\tilde{R}_w, \tilde{R}_i)}{var^{\Lambda_3}(\tilde{R}_w)} \left[\mathbb{E}^{\Lambda_3}[\tilde{R}_w] - R_f\right]. \tag{71}$$

Proof of Corollary 1

Using (20), and taking Λ_3 as given,

$$\frac{\partial}{\partial a}P = \beta\nu_3\left(-A(w_0)\mathbb{E}^{\Lambda_3}[\tilde{x}]\right) \tag{72}$$

$$\frac{\partial}{\partial \sigma}P = \beta\nu_3\left(-A(w_0)\sigma_x cov^{\Lambda_3}\left(\tilde{\omega}, \tilde{\varepsilon}\right)\right) \tag{73}$$

$$\frac{\partial}{\partial \sigma_x}P = \beta\nu_3\left(-A(w_0)\sigma cov^{\Lambda_3}(\tilde{\omega}, \tilde{\varepsilon})\right) \tag{74}$$

Note that

$$\beta^{\Lambda_3}(\tilde{x}, \tilde{w}_1) \equiv \frac{cov^{\Lambda_3}(\tilde{x}, \tilde{w}_1)}{var^{\Lambda_3}(\tilde{w}_1)} = \frac{\sigma\sigma_x cov^{\Lambda_3}(\tilde{\omega}, \tilde{\varepsilon})}{\sigma^2} = \frac{\sigma_x}{\sigma}cov^{\Lambda_3}(\tilde{\omega}, \tilde{\varepsilon}).$$

With $u'' < 0$ implying $A(w_0) > 0$, equations (28)-(27) immediately follow.

Comparative statics with the downside risk neutral measure

The following Corollary provides comparative statics of the second term in brackets on the right-hand-sides of (22) and (24) with respect to initial wealth w_0 and to expected "economic growth", holding the downside risk neutral measure constant. This allows to study changes in asset prices due to preferences about the first two degrees of risk only.

Corollary 4 *Suppose that $u'' < 0$, and let $\tilde{w}_1 \equiv w_0 + a + \tilde{\omega}$, where $\mathbb{E}^{\Lambda}[\tilde{\omega}] = 0$ under a given probability measure Λ. For given probability measure Λ,*

$$sgn\left(\frac{\partial}{\partial a}\frac{A(w_0)cov^{\Lambda}\left(\tilde{w}_1, \tilde{x}\right)}{1 - A(w_0)\mathbb{E}^{\Lambda}\left[\tilde{w}_1 - w_0\right]}\right) = sgn(cov^{\Lambda}\left(\tilde{w}_1, \tilde{x}\right)) \tag{75}$$

For any given probability measure Λ, if the utility function is CARA then

$$\frac{\partial}{\partial w_0}\frac{A(w_0)cov^{\Lambda}\left(\tilde{w}_1, \tilde{x}\right)}{1 - A(w_0)\mathbb{E}^{\Lambda}\left[\tilde{w}_1 - w_0\right]} = 0 \tag{76}$$

For any given probability measure Λ*, if the utility function is DARA then*

$$sgn\left(\frac{\partial}{\partial w_0}\frac{A(w_0)cov^{\Lambda}\left(\tilde{w}_1,\tilde{x}\right)}{1-A(w_0)\mathbb{E}^{\Lambda}\left[\tilde{w}_1-w_0\right]}\right) \quad = \quad -sgn(cov^{\Lambda}\left(\tilde{w}_1,\tilde{x}\right)) \tag{77}$$

Proof. Rewrite

$$\frac{A(w_0)cov^{\Lambda}\left(\tilde{w}_1,\tilde{x}\right)}{1-A(w_0)\mathbb{E}^{\Lambda}\left[\tilde{w}_1-w_0\right]}=\frac{cov^{\Lambda}\left(\tilde{w}_1,\tilde{x}\right)}{\frac{1}{A(w_0)}-\mathbb{E}^{\Lambda}\left[\tilde{w}_1-w_0\right]} \tag{78}$$

Then, using $\tilde{w}_1\equiv w_0+a+\tilde{\omega}$,

$$\frac{\partial}{\partial w_0}\frac{cov^{\Lambda}\left(\tilde{w}_1,\tilde{x}\right)}{\frac{1}{A(w_0)}-\mathbb{E}^{\Lambda}\left[\tilde{w}_1-w_0\right]}=\frac{A'(w_0)}{(A(w_0))^2}\frac{cov^{\Lambda}\left(\tilde{w}_1,\tilde{x}\right)}{\left(\frac{1}{A(w_0)}-\mathbb{E}^{\Lambda}\left[\tilde{w}_1-w_0\right]\right)^2} \tag{79}$$

With CARA, $A'(w_0)=0$ by definition, and the derivative in (79) is equal to zero. With DARA, $A'(w_0)<0$ by definition, and the derivative in (79) has the opposite sign of $cov^{\Lambda}\left(\tilde{w}_1,\tilde{x}\right)$.

In addition, using (78) and $\tilde{w}_1\equiv w_0+a+\tilde{\omega}$,

$$\frac{\partial}{\partial a}\frac{A(w_0)cov^{\Lambda}\left(\tilde{w}_1,\tilde{x}\right)}{1-A(w_0)\mathbb{E}^{\Lambda}\left[\tilde{w}_1-w_0\right]}=\frac{\partial}{\partial a}\frac{cov^{\Lambda}\left(\tilde{w}_1,\tilde{x}\right)}{\frac{1}{A(w_0)}-\mathbb{E}^{\Lambda}\left[\tilde{w}_1-w_0\right]}=\frac{cov^{\Lambda}\left(\tilde{w}_1,\tilde{x}\right)}{\left(\frac{1}{A(w_0)}-\mathbb{E}^{\Lambda}\left[\tilde{w}_1-w_0\right]\right)^2} \tag{80}$$

The derivative in (80) has the sign of $cov^{\Lambda}\left(\tilde{w}_1,\tilde{x}\right)$. ∎

This result has two distinct interpretations, depending on the probability measure Λ. In (24), assets are priced as if the utility function is quadratic, i.e., is risk averse but not prudent. If Λ is the physical probability measure, Corollary 4 then studies the effect of changes in economic growth and in wealth on the second term in brackets on the right-hand-side of (24), which measures the effect of risk aversion relative to the asset's expected payoff on the asset price. The second term in brackets on the right-hand-side of (22) measures the effect of risk aversion relative to the asset's expected payoff on the asset price, while the effect of higher-order risk preferences is incorporated into the change in probability measure. If Λ is the downside risk neutral probability measure ($\Lambda=\Lambda_3$), Corollary 4 then studies the effect of a change in either w_0 or a on the second term in brackets on the right-hand-side of (22), while holding the downside risk neutral probability measure constant. With decreasing absolute risk aversion, which is a natural assumption, an increase in initial wealth w_0 lowers the magnitude of the second term in brackets on the right-hand-side of (22).[14] In addition, considering that

[14]Note that the sign of this effect is in general ambiguous with the stochastic discount factor formula. With the stochastic discount factor formula, the price of an asset can be decomposed as follows: $P=\frac{\mathbb{E}[\tilde{x}]}{1+r_f}+\beta\frac{cov(u'(w_0+\alpha+\tilde{\omega}),\tilde{x})}{u'(w_0)}$. However, the sign of the derivative of the second term on the right-hand-side with respect to w_0 is ambiguous, unless strong assumptions are imposed on the utility function, such as quadratic utility. This emphasizes that separating the effect of risk aversion from the effect of higher-order risk preferences, as is possible with downside risk neutral probabilities, allows to better understand the effect of risk aversion on asset prices.

$a = \mathbb{E}^{\Lambda}[\tilde{w}_1] - w_0$, equation (75) shows that an increase in expected aggregate wealth growth increases the absolute value of this term.

Proof of Claim 2

If $u^{(4)} = 0$, then the Taylor expansion $u'(w_0) + u''(w_0)(w_s - w_0) + \frac{1}{2}u'''(w_0)(w_s - w_0)^2$ is equal to $u'(w_s)$ (which is strictly positive by assumption) for any w_s, so that $\eta_{4,s}$ as defined in (29) exists and is equal to 1 for any s.

If $u^{(4)} < 0$ (which given the assumption $u'' \leq 0$ implies $u'' < 0$), the expression $u'(w_0) + u''(w_0)(w_s - w_0) + \frac{1}{2}u'''(w_0)(w_s - w_0)^2$, which is in brackets in (29), is quadratic in w_s, decreasing in w_s at w_0 because $u'' < 0$, and is greater than $u'(w_s)$ for $w_s > w_0$ because of $u^{(4)} < 0$ (which can be seen by comparing (29) and (31)). Because $u' > 0$, this implies that the expression $u'(w_0) + u''(w_0)(w_s - w_0) + \frac{1}{2}u'''(w_0)(w_s - w_0)^2$ in brackets in (29) is strictly positive for any w_s, so that $\eta_{4,s} > 0$ for any s. In addition, with $u^{(4)} < 0$, comparing (29) and (31) shows that $\eta_{4,s} \lesseqqgtr 1$ for $w_s \gtreqqless w_0$.

The expression $1 - A(w_0)\mathbb{E}^{\Lambda_4}[\tilde{w}_1 - w_0] + \frac{1}{2}D(w_0)\mathbb{E}^{\Lambda_4}[(\tilde{w}_1 - w_0)^2]$ in (36)

We show that this expression is strictly positive.

First, consider the states s such that $u'(w_0) + u''(w_0)(w_s - w_0) + \frac{1}{2}u'''(w_0)(w_s - w_0)^2 > 0$, and therefore $\eta_{4,s} > 0$ (given (29) and $u' > 0$), which implies $\lambda_{4,s} \geq 0$. Then we have $\lambda_{4,s}\left[u'(w_0) + u''(w_0)(w_s - w_0) + \frac{1}{2}u'''(w_0)(w_s - w_0)^2\right] \geq 0$. Dividing each term by $u'(w_0)$, which is strictly positive, and summing over these states s, we have

$$\sum_{s|\eta_{4,s}>0} \lambda_{4,s}\left[1 - A(w_0)(w_s - w_0) + \frac{1}{2}D(w_0)(w_s - w_0)^2\right] \geq 0. \tag{81}$$

Second, consider the states s such that $u'(w_0) + u''(w_0)(w_s - w_0) < 0$, and therefore $\eta_{4,s} < 0$ (given (29) and $u' > 0$), which implies $\lambda_{4,s} \leq 0$. Then we have

$$\lambda_{4,s}\left[u'(w_0) + u''(w_0)(w_s - w_0) + \frac{1}{2}u'''(w_0)(w_s - w_0)^2\right] \geq 0.$$

Dividing each term by $u'(w_0)$, which is strictly positive, and summing over these states s, we have

$$\sum_{s|\eta_{4,s}<0} \lambda_{4,s}\left[1 - A(w_0)(w_s - w_0) + \frac{1}{2}D(w_0)(w_s - w_0)^2\right] \geq 0. \tag{82}$$

Third, because $\lambda_{4,s}\left[1 - A(w_0)(w_s - w_0) + \frac{1}{2}D(w_0)(w_s - w_0)^2\right] \geq 0$ for any s and $1 - A(w_0)(w_s - w_0) + \frac{1}{2}D(w_0)(w_s - w_0)^2 \neq 0$ (otherwise $\eta_{4,s}$ would not exist), both expressions on the left-hand-sides of (81) and (82) will be equal to zero only if $\lambda_{3,s} = 0$ for all s, which would imply $\sum_s \lambda_{3,s} \neq 1$, a contradiction. Therefore, at least one of the expressions on the left-hand-sides of (81) and (82) is strictly positive, while the other is nonnegative. Adding up these two sums then yields

$$\sum_s \lambda_{4,s}\left[1 - A(w_0)(w_s - w_0) + \frac{1}{2}D(w_0)(w_s - w_0)^2\right]$$

$$= 1 - A(w_0)\mathbb{E}^{\Lambda_4}\left[\tilde{w}_1 - w_0\right] + \frac{1}{2}D(w_0)\mathbb{E}^{\Lambda_4}\left[(w_s - w_0)^2\right] > 0.$$

Outer risk neutral probabilities and expected returns

Now let $\tilde{m} \equiv \frac{1}{1+r_f}\frac{g(w_0,\tilde{w}_1)}{\mathbb{E}^{\Lambda_4}[g(w_0,\tilde{w}_1)]}$. The pricing kernel $\tilde{m}$ in Proposition 2 is quadratic in $\tilde{w}_1$, and can thus be expressed as:

$$\tilde{m} = a + b\tilde{R}_w + c\tilde{R}_w^2, \tag{83}$$

where $\tilde{R}_w \equiv \frac{\tilde{w}_1}{P_w}$ is the gross return on the wealth portfolio with $t = 0$ price P_w. Equation (35) can thus be rewritten as $P = \mathbb{E}^{\Lambda_4}[\tilde{m}\tilde{x}]$. Dividing both sides by P gives $1 = \mathbb{E}^{\Lambda_4}[\tilde{m}\tilde{R}_i]$, where $\tilde{R}_i \equiv \frac{\tilde{x}}{P}$ is the gross return of asset i. In turn, this equation can be rewritten as

$$\frac{1}{\mathbb{E}^{\Lambda_4}[\tilde{m}]} = \mathbb{E}^{\Lambda_4}[\tilde{R}_i] + \frac{cov^{\Lambda_4}(\tilde{m},\tilde{R}_i)}{\mathbb{E}^{\Lambda_4}[\tilde{m}]}. \tag{84}$$

Using $\mathbb{E}^{\Lambda_4}[\tilde{m}] = \frac{1}{1+r_f} \equiv \frac{1}{R_f}$, and substituting for $\tilde{m}$, the equation above rewrites as

$$\mathbb{E}^{\Lambda_4}[\tilde{R}_i] - R_f = -\frac{b\,cov^{\Lambda_4}(\tilde{R}_w,\tilde{R}_i) + c\,cov^{\Lambda_4}(\tilde{R}_w^2,\tilde{R}_i)}{\mathbb{E}^{\Lambda_4}[\tilde{m}]}. \tag{85}$$

In particular, if asset i is the wealth portfolio with gross return $\tilde{R}_w$, we have

$$\mathbb{E}^{\Lambda_4}[\tilde{R}_w] - R_f = -\frac{b\,var^{\Lambda_4}(\tilde{R}_w) + c\,cov^{\Lambda_4}(\tilde{R}_w^2,\tilde{R}_w)}{\mathbb{E}^{\Lambda_4}[\tilde{m}]}. \tag{86}$$

It follows that there exists constants χ and ϑ independent of i such that

$$\mathbb{E}^{\Lambda_4}[\tilde{R}_i] - R_f = \chi\,cov^{\Lambda_4}(\tilde{R}_w,\tilde{R}_i) + \vartheta\,cov^{\Lambda_4}(\tilde{R}_w^2,\tilde{R}_i). \tag{87}$$

The CAPM beta

Consider a given probability measure Λ. First, using $\tilde{w}_1 \equiv w_0 + a + \sigma\tilde{\omega}$ with $var^{\Lambda}(\tilde{\omega}) = 1$,

$$\beta^{\Lambda}(\tilde{x},\tilde{w}_1) \equiv \frac{cov^{\Lambda}(\tilde{x},\tilde{w}_1)}{var^{\Lambda}(\tilde{w}_1)} = \frac{\sigma\,cov^{\Lambda}(\tilde{x},\tilde{\omega})}{\sigma^2} = \frac{cov^{\Lambda}(\tilde{x},\tilde{\omega})}{\sigma}.$$

Second, defining the asset return as $\tilde{r} \equiv \frac{\tilde{x}-P}{P}$ and the return on the wealth portfolio as $\tilde{r}_w \equiv \frac{\tilde{w}_1 - w_0}{w_0}$, the CAPM beta is:

$$\beta_{\tilde{x}} \equiv \frac{cov^{\Lambda}(\tilde{r},\tilde{r}_w)}{var^{\Lambda}(\tilde{r}_w)} = \frac{\frac{1}{Pw_0}cov^{\Lambda}(\tilde{x},\tilde{w}_1)}{\frac{\sigma^2}{w_0^2}var^{\Lambda}(\tilde{\omega})} = \frac{w_0}{P}\frac{\sigma cov^{\Lambda}(\tilde{x},\tilde{\omega})}{\sigma^2} = \frac{w_0}{P}\beta^{\Lambda}(\tilde{x},\tilde{w}_1).$$

Proof of Corollary 2

Using (34), and taking Λ_4 as given,

$$\frac{\partial}{\partial a}P = \beta\nu_4\left(-A(w_0)\mathbb{E}^{\Lambda_4}[\tilde{x}] + D(w_0)\left[a\mathbb{E}^{\Lambda_4}[\tilde{x}] + \sigma cov^{\Lambda_4}(\tilde{\omega},\tilde{x})\right]\right) \quad (88)$$

$$\frac{\partial}{\partial \sigma}P = \beta\nu_4\left(-A(w_0)cov^{\Lambda_4}(\tilde{\omega},\tilde{x}) + D(w_0)\left[\sigma E^{\Lambda_4}[\tilde{x}] + cov^{\Lambda_4}\left(a\tilde{\omega} + \sigma\tilde{\omega}^2,\tilde{x}\right)\right]\right) \quad (89)$$

$$\frac{\partial}{\partial \sigma_x}P = \beta\nu_4\left(-A(w_0)cov^{\Lambda_4}(\sigma\tilde{\omega},\tilde{\varepsilon}) + \frac{1}{2}D(w_0)cov^{\Lambda_4}\left(\sigma^2\tilde{\omega}^2 + 2a\sigma\tilde{\omega},\tilde{\varepsilon}\right)\right) \quad (90)$$

Thus, for $\mathbb{E}^{\Lambda_4}[\tilde{x}] > 0$,

$$\frac{\partial}{\partial a}P > 0 \quad \Leftrightarrow \quad \frac{A(w_0)}{D(w_0)} < \frac{a\mathbb{E}^{\Lambda_4}[\tilde{x}] + \sigma cov^{\Lambda_4}(\tilde{\omega},\tilde{x})}{\mathbb{E}^{\Lambda_4}[\tilde{x}]} = a + \sigma\frac{cov^{\Lambda_4}(\tilde{\omega},\tilde{x})}{\mathbb{E}^{\Lambda_4}[\tilde{x}]},$$

and for $\mathbb{E}^{\Lambda_4}[\tilde{x}] < 0$, the inequality is reversed. For $\beta^{\Lambda_4}(\tilde{x},\tilde{w}_1) > 0$,

$$\frac{\partial}{\partial \sigma}P > 0 \quad \Leftrightarrow \quad \frac{A(w_0)}{D(w_0)} < \frac{\sigma E^{\Lambda_4}[\tilde{x}] + cov^{\Lambda_4}(a\tilde{\omega} + \sigma\tilde{\omega}^2,\tilde{x})}{cov^{\Lambda_4}(\tilde{\omega},\tilde{x})} = a + \sigma\frac{E^{\Lambda_4}[\tilde{x}]}{cov^{\Lambda_4}(\tilde{\omega},\tilde{x})} + \sigma\frac{cov^{\Lambda_4}(\tilde{\omega}^2,\tilde{x})}{cov^{\Lambda_4}(\tilde{\omega},\tilde{x})}$$

$$\frac{\partial}{\partial \sigma_x}P > 0 \quad \Leftrightarrow \quad \frac{A(w_0)}{D(w_0)} < \frac{cov^{\Lambda_4}(0.5\sigma^2\tilde{\omega}^2 + a\sigma\tilde{\omega},\tilde{\varepsilon})}{cov^{\Lambda_4}(\sigma\tilde{\omega},\tilde{\varepsilon})} = a + \frac{\sigma}{2}\frac{cov^{\Lambda_4}(\tilde{\omega}^2,\tilde{x})}{cov^{\Lambda_4}(\tilde{\omega},\tilde{x})},$$

and for $\beta^{\Lambda_4}(\tilde{x},\tilde{w}_1) < 0$, the inequalities are reversed. But $\frac{A(w_0)}{D(w_0)} = -\frac{u''(w_0)}{u'''(w_0)} \equiv \frac{1}{P(w_0)}$. Letting $\beta^{\Lambda_4}(\tilde{x},\tilde{w}_1) \equiv \frac{cov^{\Lambda_4}(\tilde{w}_1,\tilde{x})}{var^{\Lambda_4}(\tilde{w}_1)} = \frac{cov^{\Lambda_4}(\tilde{\omega},\tilde{x})}{\sigma}$ be the beta of the asset price with respect to aggregate wealth under the outer risk neutral measure, this gives equations (43)-(48).

Proof of Corollary 3

First of all,

$$cov^{\Lambda_4}\left((\tilde{w}_1 - w_0)^2,\tilde{x}\right) = cov^{\Lambda_4}\left((a + \sigma\tilde{\omega})^2,\tilde{x}\right) = \sigma cov^{\Lambda_4}\left(2a\tilde{\omega} + \sigma\tilde{\omega}^2,\tilde{x}\right). \quad (91)$$

In addition, for $i \in \{1,2\}$,

$$cov^{\Lambda_4}(\tilde{\omega}^i,\tilde{x}) = \sigma_x cov^{\Lambda_4}(\tilde{\omega}^i,\tilde{\varepsilon}) = \sigma_x\rho_{\omega^i,\varepsilon},$$

where $\rho_{\omega^i,\varepsilon}$ is the coefficient of correlation between $\tilde{\omega}^i$ and $\tilde{\varepsilon}$. Then, given $cov^{\Lambda_4}(\tilde{\omega},\tilde{\varepsilon}) \neq 0$,

$$\frac{cov^{\Lambda_4}((\tilde{w}_1 - w_0)^2,\tilde{x})}{cov^{\Lambda_4}(\tilde{w}_1,\tilde{x})} = 2a + \sigma\frac{cov^{\Lambda_4}(\tilde{\omega}^2,\tilde{x})}{cov^{\Lambda_4}(\tilde{\omega},\tilde{x})} = 2a + \sigma\frac{\rho_{\omega^2,\varepsilon}}{\rho_{\omega,\varepsilon}} \longrightarrow_{a\to 0,\sigma\to 0} 0. \quad (92)$$