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Integrating Real and Financial Decisions of the Firm

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Abstract:
We study the issue of integrating real and financial decisions in a monopoly firm with risk-averse decision-makers. To that end, we combine the decisions of the firm and of the shareholders in a very simple but robust model, with uncertainty in the real market and CARA preferences. We show the existence of equilibrium either in a competitive and a uncompetitive financial market, though different assumptions are needed in each case. In all situations, access to the financial market leads to risk-sharing and an increase in production, but only the competitive case is Pareto optimal. When either the firm or the outside investors act as leaders, the optimal risk-sharing is distorted to favor the leader. We also discuss the effect that changes on the coefficients of risk aversion have on the equilibrium outcomes.

Keywords: Existence of equilibrium, Financial sector, Firm behavior, Market power, Monopoly, Nash equilibrium, Perfect competition, Publicly-traded firm, Risk aversion, Risk taking, Shareholder behavior, Stackelberg equilibrium

JEL Classification: D21, D42, D82, D83, D84, L12, L15

1 Introduction

Uncertain and risky events are ubiquitous in society. While economic agents cannot eliminate all of the exogenous source of risk, they can exercise a certain control over the amount of risk they face through the market process.[1] Specifically, markets and prices allocate resources to different risky activities, and among different agents. For instance, when a firm undertakes a risky project in the real sector, the size of the project as well as the share of risk borne by each shareholder depend on market forces in both the real and financial sectors. In particular, the choice and allocation of risk depend on the prices of goods in the real sector as well as the prices of financial instruments. These prices depend, in turn, on the preferences of agents, the alternative assets of the shareholders, the market structure, and the exogenous source of risk.

Yet, in the standard framework of industrial organization, markets and prices play no role in determining which types of risk are undertaken by firms and which groups of agents bear the risk. Rather, risk vanishes under the postulate that firms maximize expected profit, even if their shareholders are risk-averse. The risk-neutrality of firms owned by risk-averse shareholders is generally justified on the grounds that the shareholders' portfolios of assets are well-diversified, to the point of eliminating any exposure to, and concern for risk.[2] In other words, while the shareholders are risk-averse, portfolio diversification induces their firms to *act* as risk-neutral, and, thus, to maximize expected profit.[3]

There are two main issues with this justification. First, the market process by which shareholders diversify their portfolio is not modeled. The diversifi-

[1]The reduction of risk comes at a cost, so that, even if feasible, an economic agent would not necessarily eliminate risk all together.

[2]See Tirole (1988, pp. 34-35), including footnote 61, and Salanié (1997, p. 53).

[3]Another argument in support of the risk-neutrality of firms is that shareholders are risk-neutral. If risk-averse agents owned shares of a risky asset, they could benefit from an arbitrage opportunity as well as rid themselves of any exposure to risk by selling their shares to risk-neutral agents. Hence, all risky assets would be owned by risk-neutral shareholders. Risk-neutral shareholders would invest all of their wealth in the asset with the highest expected return, and, thus, all assets would have the same rate of return, which is inconsistent with observed behavior.

cation of assets is costly and might not benefit all shareholders in the same way. Moreover, the interplay between the shareholders' portfolio selection and the firms' decisions is an important link. On the one hand, the interaction of the shareholders in the financial market influence the behavior of the firms in the real market, which is not necessarily one of maximizing expected profit. On the other hand, the allocation of risk through the financial market depends on the distribution of real profit, which, in turn, depends on the decisions of the firms. Second, the high variability of all market indicators makes it difficult to believe that portfolio diversification renders shareholders immune to risk. Indeed, the allocation of wealth among many assets only reduces, but cannot eliminate, the unsystematic risk that emanates from each risky asset. Moreover, systematic risk remains and affects the payoffs of all assets. Thus, despite the availability of a wide range of financial instruments, shareholders must accept risk.

It is a purpose of this paper to provide a conceptual framework that links the risky production decisions of the firm with the risk-sharing financial decisions of the investors. From a financial point of view, this is equivalent to studying the influence of markets and prices on the choice and allocation of a risky asset issued by a firm. To that end, we embed a mean-variance approach to the shareholders' portfolio selection, pioneered by Markowitz (1952) and Tobin (1958), into the theory of the firm. By establishing an explicit link between the behavior of the shareholders and the firm's, the real and financial sectors are integrated. In particular, the payoff of the risky asset depends on the level of output, and reflects the uncertainty that emanates from the real sector.

We consider a monopoly initially owned by an entrepreneur (the managing shareholder) who has the ability to float shares of a risky asset (tied to the random profit of the monopoly). In our model, the deciding shareholder of a firm, called the entrepreneur, undertakes a risky project in the real sector and interacts with the remaining shareholder, called the investor, in the financial sector. The project is risky because the firm faces a random price in the real market. The allocation of risk among risk-averse shareholders is achieved by selling shares of a risky asset in the financial market. Shares of the risky asset define the ownership structure of the firm and represent claims to the profit derived in the real sector. While the entrepreneur allocates the profit of the firm among the shareholders, the entrepreneur retains control of the firm's decisions. Specifically, the entrepreneur decides both the level of output and the ownership structure of the firm.

To provide a bench-marking Pareto-optimal solution, we begin by studying the Nash equilibrium with a competitive financial market. Financial access, i.e, floating part of the shares, leads to the global acceptance of more risk and, hence, to an increase of equilibrium output. Under the assumptions made, the limits of this increase in output, as the fraction of stock floated tends to 1, equals the classical monopoly solution, with or without risk depending on the risk-aversion of the investor.

We then consider the Stackelberg equilibrium under two scenarios. In the first one, the entrepreneur is the leader (sophisticated agent) whereas the investor is the follower (naive agent). In the second one, we reverse roles by having the investor as the leader. These scenarios lead to the basic results stated above, but differ about the particular optimal solutions. In all scenarios financial access lead to an increase in output by the firm, due to better risk-sharing between the entrepreneur and the outside investor. This is one of the main result of this work.

A second important result is that only the competitive equilibrium is Pareto efficient; both Stackelberg equilibrium lead to smaller output levels than the competitive Nash equilibrium. In the competitive market, the fraction of shares allocated to each agent is directly proportional to their respective coefficient of risk aversion; when any of the agents assumes a leading role, that fraction is distorted to favor the leader goals.

There are also differences on how the fraction of shares floated and the equilibrium output change with the risk aversion coefficients. The fraction of shares floated always increases when risk aversion of the entrepreneur also increases and decreases when risk aversion of the investor increases. In the competitive market and when the entrepreneur is leading, that fraction varies from zero to 1 when the risk aversion coefficient of the entrepreneur goes from zero to infinity (or the risk aversion coefficient of the investor goes from infinity to zero). But when the leader is the investor, the fraction of shares sold varies from zero to one half; the investor never demands more than half the shares in order to depress the financial price.

The relationship between risk and firm behavior has been present in the literature for some decades. Baron (1970), Baron (1971), Sandmo (1971), and Leland (1972) studied the impact of risk aversion on the decisions of a risk averse firm in a competitive and in an imperfectly competitive market. However, these early works made no attempt to relate behavior of the firm with its ownership structure or the functioning of the financial market. Later works have established a relationship between real and financial sectors: Dotan and Ravid (1985), Prezas (1988), Brander and Lewis (1986) and Showalter (1995), arrived there while studying the problem of optimal debt-equity allocation; Jain and Mirman (2000) work on insider trading also shows that both sectors are related.

Mirman and Santugini (2013) on risk-sharing and financial markets goes much further. They analyze a model with a risk-averse owner of a monopolist firm (the entrepreneur) facing the option of selling part of the stock of his firm to a risk-averse outside investor. The entrepreneur retains control over all decisions of the firm, notably on the quantity of output supplied in the real market. To optimize his utility this entrepreneur must take into account, simultaneously, his decisions on the real and on the financial market, because his final expected wealth depends on both. This dual perspective distinguishes this model from most of the previous literature and integrates real and financial equilibrium.

The paper is organized as follows. After this introduction, Section 2 studies the Nash equilibrium with a competitive financial market, whereas Section 3 considers the Stackelberg equilibrium. We provide concluding remarks in Section 4.

2 Nash Equilibrium with Competitive Financial Market

In this section, we present a general model combining the behavior of the firm (in the real and financial sectors) and the behavior of the shareholders. We establish conditions under which there exists a Nash equilibrium with a competitive financial market, then we characterize and discuss this equilibrium.

2.1 Set Up

Consider a firm that is a monopoly in a real market and has access to the financial market.[4] In the real market, the firm faces a random demand with known distribution and chooses the level of output $q \geq 0$. Specifically, the random price corresponding to supplying q units is $\tilde{p}_R = P_R(q) + \tilde{\varepsilon}$ where $P_R(q)$ is the expected inverse demand and $\tilde{\varepsilon}$ is a normally-distributed shock.[5]

Assumption 2.1. *$\tilde{\varepsilon} \sim N(0, \sigma^2)$.*

The random profit of the firm is thus $\pi(q, \tilde{\varepsilon}) = (P_R(q) + \tilde{\varepsilon})q$. The expected profit is assumed to be strictly concave in the level of output.

Assumption 2.2. *$P_R''(q)q + 2P_R'(q) < 0$.*

In the financial sector, the firm issues $S \in \Omega_S \subseteq \Re_+$ equity shares.[6] Each share is a claim of $\frac{1}{S}$ of the total profit so that each share receives a random payoff $\pi(q, \tilde{\varepsilon})/S$. In addition to choosing the total number of shares, the firm decides on the fraction $1 - \omega \in [0, 1]$ of the shares to be sold in the financial market at unit price p_F.[7] Hence, the variable ω defines the

[4] The adjective *real* refers to the sector of goods and services other than those of financial nature.

[5] The subscript R refers to the *real sector* and the tilde sign differentiates a random variable from its realization.

[6] The type of restriction imposed on the set Ω_S turns out to be key for the existence of the equilibrium and the comparative analysis.

[7] The subscript F refers to the *financial sector*.

ownership structure of the firm, which specifies the allocation of the random profit among the shareholders.

The objective of each shareholder is to maximize the expected utility of final wealth. Each shareholder diversifies wealth between the risky asset issued by the firm and a risk-free asset. Without loss of generality, we assume that there are only two shareholders, i.e., an entrepreneur and an investor. The entrepreneur is the founder of the firm and the original claimant of the profit generated by his entrepreneurial prospects. The entrepreneur is also the managing shareholder of the firm, making the output decision, issuing the total number of shares, and deciding on the number of shares to be floated. Having no initial wealth, the entrepreneur's random final wealth is

$$\widetilde{W}'_E = \omega \cdot \pi(q, \tilde{\varepsilon}) + p_F \cdot (1 - \omega) \cdot S \tag{1}$$

where $\omega \cdot \pi(q, \tilde{\varepsilon})$ is the entrepreneur's portion of the random profit of the firm and $p_F \cdot (1 - \omega) \cdot S$ is the wealth generated from selling $(1 - \omega) \cdot S$ shares at unit price p_F, and investing $p_F \cdot (1 - \omega) \cdot S$ in a risk-free asset with a rate of return normalized to one.

Unlike the entrepreneur, the investor does not have entrepreneurial prospects and has no direct control over the decisions of the firm. The investor uses his initial wealth $W_I > 0$ to purchase shares of the risky asset and the risk-free asset. Hence, the investor's random final wealth is

$$\widetilde{W}'_I = W_I + \pi(q, \tilde{\varepsilon})z/S - p_F z \tag{2}$$

where z is the number of shares purchased by the investor. Here, $W_I - p_F z$ is invested in the risk-free asset and $\pi(q, \tilde{\varepsilon})z/S$ is the random payoff corresponding to z shares of the risky asset. Note that the return on a share of the firm is $\pi(q, \tilde{\varepsilon})/S - p_F$.

Each shareholder maximizes the expected utility of final wealth defined by (1) or (2). The shareholders are assumed to be risk-averse in final wealth with constant absolute risk aversion (CARA).

Assumption 2.3. *The coefficients of absolute risk aversion are* $a_E > 0$ *and*

$a_I > 0$ *for the entrepreneur and the investor, respectively.*[8]

From (1), given that $\tilde{p}_R = P_R(q) + \tilde{\varepsilon}$, the certainty equivalent of the entrepreneur is[9]

$$CE_E = \omega \cdot P_R(q)q + p_F \cdot (1-\omega) \cdot S - a_E\sigma^2\omega^2q^2/2. \tag{3}$$

Here, $\omega \cdot P_R(q)q + p_F \cdot (1-\omega) \cdot S$ is the expected payoff to the entrepreneur from the real and financial sectors weighted by the level of ownership. The term $a_E\sigma^2\omega^2q^2/2$ is the risk premium of the entrepreneur. The risk premium plays the role of a cost, due to risk aversion, imposed on the entrepreneur for bearing part of the risk. From (2), the certainty equivalent of the investor is

$$CE_I = W_I + (P_R(q)q/S - p_F)z - a_I\sigma^2(q/S)^2z^2/2 \tag{4}$$

where $W_I + (P_R(q)q/S - p_F)z$ is the expected mean of final wealth and $a_I\sigma^2(q/S)^2z^2/2$ is the risk premium.

2.2 Equilibrium

Having described the model, we now define the Nash equilibrium with a competitive financial market. The entrepreneur and the investor move simultaneously in a Nash equilibrium. The financial sector is perfectly competitive, i.e., the financial price is given, and, thus, neither the entrepreneur nor the investor can take into account the effect of their decisions on the financial price.

To ensure the existence of a Nash equilibrium in this competitive market the set Ω_S must be restricted. The reason is that, if the financial price is given, the firm has an incentive to increase the total number of shares to infinity, which yields no solution for the managing shareholder. Therefore,

[8] In other words, utility functions for final wealth x are exponential: $u(x; a) = -e^{-ax}, a \in \{a_E, a_I\}$.

[9] The expected utility of the entrepreneur is $\mathbb{E}u(\widetilde{W_E}; a_E) = -e^{-a_E CE_E}$, where $\mathbb{E}$ is the expectation operator.

we make the assumption [10] that the total number of shares issued is fixed exogenously, i.e., $\Omega_S = \{S|S = \overline{S} \in \Re_+\}$.

Some constraint on Ω_S in a competitive market conforms to the perceived behavior of agents on the financial market: investors would never accept to pay a given price for an asset whose supply can increase indefinitely, debasing its value; conversely, no rational issuer expects to sell, at a given price, an unbounded number of claims to a limited profit. Exogenizing the total number of shares issued reflects the frequent fact that the firm first decides and publicizes the total number of shares in which its capital is divided and then, based on that information, decides the other conditions of an IPO. In other words, what the firm really decides is the percentage of capital to float. Also, though this competitive scenario is mostly interesting as a bench-mark, we can think of situations in the financial market where both the firm and the outside investors might behave as price-takers[11].

In equilibrium, the price of the risky asset clears the financial market by equating the quantity demanded by the investor with the quantity supplied by the firm (or the entrepreneur). The equilibrium consists of the firm's decisions made by the entrepreneur $\left\{q^*, \omega^* | S = \overline{S}\right\}$, the investor's amount of shares of the risky asset z^*, and the financial price p_F^*. The entrepreneur's decisions $\left\{q^*, \omega^* | S = \overline{S}\right\}$ have a direct effect on the investor's payoffs. However, the investor's decision has no influence on the entrepreneur's payoffs. Both shareholders are affected indirectly by each other through the financial price.

Definition 2.4. *The tuple* $\left\{q^*, \omega^*, z^*, p_F^* | S = \overline{S}\right\}$ *is a Nash equilibrium with*

[10]In Mirman and Santugini (2013) the restriction considered was to equate the total number of shares to output so that each share is a claim to the profit of one unit of output. This assumption retains the idea that the firm wishes to increase the number of shares to increase the proceeds from the financial market. At the same time, it allows for the existence of an equilibrium. Indeed, the total number of shares cannot go to infinity because, being equal to output, it is limited by the real demand function. With this restriction on Ω_S we obtain some of the results of the present paper, but not others.

[11]When a investment bank intermediates an IPO and guaranties its success, it is normal that the intermediary evaluates the market conditions and determines the price that ensures the success of the placement. The issuer then decides the percentage of shares to be floated at the proposed price, while the outside investors place their orders and determine the volume demanded.

a competitive financial market if

1. *Given* $\overline{S}, q^*$ *and* p_F^**, the investor's quantity demanded for the risky asset is*

$$z^* = \arg\max_{z\geq 0} \left\{ W_I + (P_R(q^*)q^*/\overline{S} - p_F^*)z - a_I\sigma^2(q^*/\overline{S})^2 z^2/2 \right\}. \quad (5)$$

2. *Given* p_F^**, subject to* $q \geq 0, \omega \in [0,1], S = \overline{S}$*,*

$$\{q^*, \omega^*\} = \arg\max_{q,\omega} \left\{ \omega \cdot P_R(q)q + p_F^* \cdot (1-\omega) \cdot \overline{S} - a_E\sigma^2\omega^2 q^2/2 \right\}. \quad (6)$$

3. *Given* $\{\omega^*, z^*, \overline{S}\}$*,* $p_F^* > 0$ *satisfies the market-clearing condition* $z^* = (1-\omega^*)\overline{S}$*.*

Proposition 2.5 characterizes the equilibrium when the total number of shares is fixed. Hence, the firm chooses output and ownership, i.e., $\{q, \omega\}$.[12]

Proposition 2.5. *Suppose that* $\Omega_S = \{S | S = \overline{S} \in \Re_+\}$*. Then, there exists a Nash equilibrium with a competitive financial market. In equilibrium, output* q^* *satisfies*

$$P_R'(q^*)q^* + P_R(q^*) = \omega^* a_E \sigma^2 q^*, \quad (7)$$

the allocation of risk is defined by

$$\omega^* = \frac{a_I}{a_I + a_E}, \quad (8)$$

and $S^* = \overline{S}$*. Moreover, the investor's quantity demanded is*

$$z^* = \frac{P_R(q^*)q^*/\overline{S} - p_F^*}{a_I\sigma^2(q^*/\overline{S})^2} \quad (9)$$

and the financial price is

$$p_F^* = P_R(q^*)q^*/\overline{S} - (1-\omega^*)a_I\sigma^2 q^{*2}/\overline{S}. \quad (10)$$

[12] Alternatively, setting an exogenous upper bound on the number of shares also yields existence, i.e., $\Omega_S = [0, \overline{S}], \overline{S} \in (0, \infty)$.

Proof. Given Ω_S, $S^* = \overline{S}$. The first-order condition corresponding to (5) evaluated at $S^* = \overline{S}$ yields (9). Next, plugging (9) and $S^* = \overline{S}$ into the market-clearing equilibrium $z^* = (1 - \omega^*)S^*$ yields (10). Finally, the first-order conditions corresponding to (6) evaluated at $S = \overline{S}$ are

$$q : \omega \cdot [P_R'(q)q + P_R(q)] - \omega^2 a_E \sigma^2 q = 0, \tag{11}$$

$$\omega : P_R(q)q - p_F^* \overline{S} - a_E \sigma^2 \omega q^2 = 0, \tag{12}$$

evaluated at $q = q^*$ and $\omega = \omega^*$. Rearranging (11) yields (7). Plugging (10) into (12) and solving for ω^* yields (8). □

2.3 Discussion

Having characterized the equilibrium, we begin by noting that the restriction for Ω_S has no effect on the allocation of risk. Indeed, from (8), the fraction of shares to be floated is independent of the choice of Ω_S and depends only on the relative size of the risk aversion coefficients. If $a_E/a_I \to 0$, then $\omega^* \to 1$ and the entrepreneur bears all the risk, i.e., no floating. If $a_E/a_I \to \infty$, then $\omega^* \to 0$ and the investor bears all the risk, i.e., 100% floating. The equilibrium output under (7) is Pareto optimal. Indeed, substituting (8) into (7) and rearranging the left-hand side yields

$$P_R'(q^*)q^* + P_R(q^*) = a_E \sigma^2 \omega^2 q^* + a_I \sigma^2 \cdot (1 - \omega)^2 q^*. \tag{13}$$

Result (13) indicates that the total marginal revenue of output (the left-hand side of (13)) is equal to the total marginal cost of risk for *both* agents (the right-hand side (13)).[13]

Next, the restriction on Ω_S has no effect on most of the comparative analysis for output. Indeed, the direction of the effect of an increase in a_I or σ^2 is independent of the choice of Ω_S. Specifically, a more risk-averse investor induces the firm to decrease production, i.e., $\partial q^*/\partial a_I < 0$. Similarly, regardless of the choice of Ω_S, an increase in the variance of the shock increases

[13] See Appendix A.

the marginal cost of bearing some risk, i.e., it increases the right-hand side of (7). This induces the firm to decrease output, i.e., $\partial q^*/\partial\sigma^2 < 0$.

The effect of the entrepreneur's risk aversion on the level of output is negative, i.e., $\partial q^*/\partial a_E < 0$. Although shareholders have an aversion for risk, their rewards (expected return) depend positively on the amount of risk the firm takes. In other words, the higher the risk premium of an investor, the higher the premium (in terms of expected returns) given to a shareholder to bear part of the risk of the firm. This conflict between shareholders disdain for risk and the increase in the payment when risk increases is important. For $\Omega_S = \{S|S = \overline{S} \in \Re_+\}$, the entrepreneur decreases output so that the firm takes on less risk, i.e., $\partial q^*/\partial a_E < 0$ implies that $\partial \mathbb{V}\pi_R(q^*, \tilde{\varepsilon})/\partial a_E < 0$[14]. That is, with a fixed total number of shares, the entrepreneur only reacts to an increase in his risk aversion but has no concern for encouraging the investor to take on more risk.

We conclude this discussion by looking at the limiting cases. As $\sigma^2 \to 0$, the level of output equals the solution for a risk-averse monopoly facing no risk, i.e., $P_R'(q^*)q^* + P_R(q^*) = 0$. As $a_I = 0$, $\omega^* = 0$ and the level of output tends to the solution for a monopoly facing no risk, i.e., $P_R'(q^*)q^* + P_R(q^*) = 0$. As $a_I \to \infty$, $\omega^* \to 1$ and the level of output tends to the solution of a monopoly owned solely by a risk-averse entrepreneur, i.e., $P_R'(q^*)q^* + P_R(q^*) = a_E\sigma^2 q^*$. This is shown in Figure 1.[15]

As $a_E = 0$, $\omega^* = 1$ and the level of output equals the solution for a risk-averse monopoly facing no risk, i.e., $P_R'(q^*)q^* + P_R(q^*) = 0$. As $a_E \to \infty$, $\omega^* \to 0$ the level of output tends to the solution of a risk-averse monopoly owned by the investor who takes on all the risk, i.e., $P_R(q^*) = a_I\sigma^2 q^*$. See Figure 2.

[14] Letting $\mathbb{V}$ be the variance operator, $\mathbb{V}\pi_R(q^*, \tilde{\varepsilon}) = \sigma^2 q^{*2}$ reflects the degree to which the entrepreneur takes risk on behalf of the firm.

[15] In the following graphs we assumed, for simplicity, that real demand is linear.

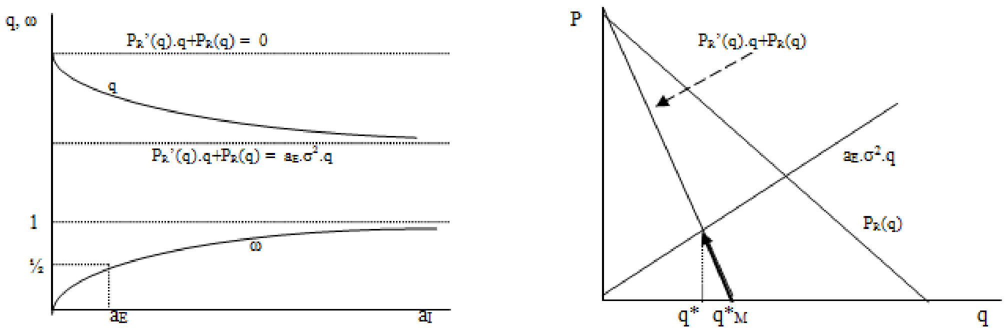

Figure 1: The Effect of a_I on q^* and ω^* when $\Omega_S = \{S | S = \overline{S} \in \Re_+\}$.

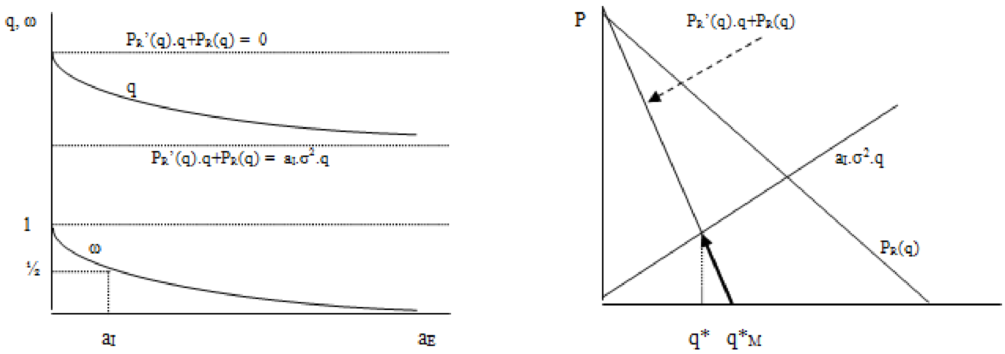

Figure 2: The Effect of a_E on q^* and ω^* when $\Omega_S = \{S | S = \overline{S} \in \Re_+\}$

3 Stackelberg Equilibrium with Non Competitive Financial Market

In this section, we consider the Stackelberg equilibrium in which one agent is sophisticated whereas the other agent is naive. In other words, one of the agents is a leader and the other one is a follower. The Stackelberg environment is not compatible with a competitive financial market. Hence, the leader (whether the entrepreneur or the investor) has market power in the financial sector. We begin by showing that there exists a Stackelberg equilibrium with a leading entrepreneur without the need to impose any restrictions on the Ω_S. However, the total number of shares and the financial price remain undetermined. i.e., one depends on the arbitrary choice of the other. We then show that there is no Stackelberg equilibrium with a leading investor unless restrictions are imposed on the Ω_S.

3.1 Leading Entrepreneur

We first define the Stackelberg equilibrium with the entrepreneur as the leader. This is a situation that may arise in many IPOs, as the firm may be sophisticated enough to determine itself the floating price, thus conditioning the outside investors' demand.

Definition 3.1. *The tuple $\{q^*, \omega^*, S^*, z^*(q^*, \omega^*, S^*), p_F^*\}$ is a Stackelberg equilibrium (leading entrepreneur) with a non-competitive financial market if*

1. *Given $\{q^*, \omega^*, S^*\}$ and p_F^*, the investor's quantity demanded for the risky asset is*

$$z^*(q^*, \omega^*, S^*) = \arg\max_{z\geq 0} \left\{W_I + (P_R(q^*)q^*/S^* - p_F^*)z - a_I\sigma^2(q^*/S^*)^2 z^2/2\right\}. \tag{14}$$

2. *Given $z^*(q, \omega, S)$, subject to $q \geq 0, \omega \in [0,1], S \in \Omega_S$,*

$$\{q^*, \omega^*, S^*\} = \arg\max_{q,\omega,S} \left\{\omega P_R(q)q + D^*(q, \omega, S) \cdot (1-\omega) \cdot S - a_E\sigma^2\omega^2 q^2/2\right\} \tag{15}$$

   where $p_F = D^*(q, \omega, S)$ is the inverse financial demand defined by $z^*(q, \omega, S) = (1-\omega)S$.

3. *Given $\{q^*, \omega^*, S^*, z^*(q^*, \omega^*, S^*)\}$, $p_F^* > 0$ satisfies the market-clearing condition $z^*(q^*, \omega^*, S^*) = (1-\omega^*)S^*$.*

Proposition 3.2 states that there exists an equilibrium when the entrepreneur is the leader. Hence, another way to reestablish existence without any restriction on the total number of shares is to assume that the entrepreneur has market power in the financial sector. However, the total number of shares and the financial price cannot be uniquely and independently determined. The reason is that the inverse financial demand is inversely proportional to S so that, from (15), the total number of shares has no effect on the entrepreneur's certainty equivalent. In other words, the relevant variable decided by the entrepreneur is the total asking value of the firm.

Proposition 3.2. *Suppose that $\Omega_S = \Re_+$. Then, there exists a Stackelberg equilibrium with a sophisticated entrepreneur. In equilibrium, output q^* satisfies*

$$P_R'(q^*)q^* + P_R(q^*) = \omega^* a_E \sigma^2 q^*, \tag{16}$$

the allocation of risk is defined by

$$\omega^* = \frac{2a_I}{2a_I + a_E}. \tag{17}$$

Moreover, the investor's quantity demanded is

$$z^*(q^*, \omega^*, S^*) = \frac{P_R(q^*)q^*/S^* - p_F^*}{a_I \sigma^2 (q^*/S^*)^2}. \tag{18}$$

and

$$S^* p_F^* = P_R(q^*)q^* - (1-\omega^*)a_I \sigma^2 q^{*2} \tag{19}$$

where the total number of shares and the financial price cannot be determined separately.

Proof. The first-order condition corresponding to (14) yields (18). Next, plugging (18) (for any q, ω, and S) into $z^*(q, \omega, S) = (1-\omega)S$ and solving for the inverse financial demand function yields

$$D^*(q, \omega, S) = P_R(q)q/S - (1-\omega)a_I \sigma^2 q^2/S. \tag{20}$$

Plugging (20) into (15) yields the entrepreneur's maximization problem

$$\max_{q,\omega} \left\{ P_R(q)q - (1-\omega)^2 a_I \sigma^2 q^2 - a_E \sigma^2 \omega^2 q^2/2 \right\} \tag{21}$$

where S has no effect on the entrepreneur's certainty equivalent. From (19), it follows that

$$p_F^* = \left(P_R(q^*)(q^*) - a_I\sigma^2\omega q^{*2}\right)/S^* \tag{22}$$

S^* and thus p_F^* are undefined. The first-order condition corresponding to (21) are

$$q : P_R'(q)q + P_R(q) - 2(1-\omega)^2 a_I\sigma^2 q - a_E\sigma^2\omega^2 q = 0, \tag{23}$$
$$\omega : 2(1-\omega)a_I\sigma^2 q^2 - a_E\sigma^2\omega q^2 = 0, \tag{24}$$

evaluated at $q = q^*$ and $\omega = \omega^*$. Solving (24) for ω^* yields (17). Plugging (17) into (23) and rearranging yields (16). Plugging q^* and ω^* into (20) and multiplying by S^* yields (19). □

Having characterized the Stackelberg equilibrium with leading entrepreneur, we now compare equilibrium values under Nash and Stackelberg. First, comparing (8) with (17), it follows that the entrepreneur shares less risk under Stackelberg than under Nash (regardless of the restriction on Ω_S for Nash). This is due to the fact that the investor's coefficient of risk aversion is weighed twice under Stackelberg. Under Stackelberg, the entrepreneur takes into account the effect of an increase in shares offered in the financial price through the marginal risk cost of the investor. Hence, the entrepreneur sets a smaller float of shares in order to increase the financial price.

The firm's output is thus lower under Stackelberg than under Nash because the right-hand side in (16) is now bigger than in (7)[16]. The signs of the effects of the risk coefficients on ω^* and q^*, as well as the limits of ω^* (when the risk coefficients approach zero or infinity) remain unchanged. The limits of q^* when a_E or a_I tend to zero or when a_I tends to infinity are also left unchanged. However, under Stackelberg, when a_E tends to infinity, $\omega^* \to 0$ and output q^* satisfies $P_R'(q^*)q^* + P_R(q^*) = 2a_I\sigma^2 q^*$ in the limit.

[16]From expression (16), it can be shown that the equilibrium output is no longer Pareto optimal.

3.2 Leading Investor

Having considered the case of a leading entrepreneur, we now study the Stackelberg equilibrium with a leading investor. This scenario can arise in situations when the initiative of the financial floating comes from the outside investor, for instance, when a multinational makes an offer to buy a qualified position in a smaller private firm.

Definition 3.3. *The tuple* $\{q^*(z^*), \omega^*(z^*), S^*(z^*), z^*, p_F^*\}$ *is a Stackelberg equilibrium (leading investor) with a non-competitive financial market if*

1. *Given* $\{q^*(z), \omega^*(z), S^*(z)\}$*, the investor's quantity demanded for the risky asset is*

$$z^* = \arg\max_{z\geq 0} \left\{ W_I + (P_R(q^*(z))q^*(z)/S^*(z) - D^*(z))z - a_I\sigma^2(z/S^*(z))^2/2 \right\} \tag{25}$$

where $p_F = D^*(z)$ *is the inverse financial demand defined by* $z = (1 - \omega(z))S(z)$.

2. *Given* p_F^**, subject to* $q \geq 0, \omega \in [0,1], S \in \Omega_S$*,*

$$\{q^*(z), \omega^*(z), S^*(z)\} = \arg\max_{q,\omega,S} \left\{ \omega P_R(q)q + p_F^* \cdot (1-\omega) \cdot S - a_E\sigma^2\omega^2 q^2/2 \right\} \tag{26}$$

3. *Given* $\{q^*(z^*), \omega^*(z^*), S^*(z^*), z^*\}$*,* $p_F^* > 0$ *satisfies the market-clearing condition* $z^* = (1 - \omega^*(z^*))S^*(z)$.

Proposition 3.4. *Suppose that* $\Omega_S = \Re_+$*. Then, there exists no Stackelberg equilibrium with a leading investor.*

Proof. From (26), given $p_F^* > 0$, there is no solution for S^*. □

In order to obtain an equilibrium for 3.3 with a leading investor, we must guaranty there is a solution for the entrepreneur's optimization problem with a given price. As we have seen before, this requires some kind of constraint on S. Again, the rational is that no outside investor will make an offer for a number of shares of a target firm without knowing the total number of shares outstanding. An offer is made for a percentage of the total capital, not for a fixed number of shares in an undetermined total.

Proposition 3.5 provides the equilibrium values under Stackelberg when the total number of shares is set exogenously. In equilibrium, the decisions of the entrepreneur do not depend on z directly. Hence, notation is simplified by writing $\{q^*, \omega^*\}$.

Proposition 3.5. *Suppose that $\Omega_S = \{S | S = \overline{S} \in \Re_+\}$. Then, there exists a Stackelberg equilibrium with a leading investor. In equilibrium, output q^* satisfies*

$$P_R'(q^*)q^* + P_R(q^*) = \omega^* a_E \sigma^2 q^*, \tag{27}$$

the allocation of risk is defined by

$$\omega^* = \frac{a_I + a_E}{a_I + 2a_E}, \tag{28}$$

and $S^* = \overline{S}$. Moreover, the investor's quantity demanded is

$$z^* = \frac{a_E \overline{S}}{2a_E + a_I}, \tag{29}$$

and the financial price is

$$p_F^* = P_R(q^*)q^*/\overline{S} - \frac{a_E + a_I}{2a_E + a_I} a_E \sigma^2 q^{*2}/\overline{S}. \tag{30}$$

Proof. Given $S^*(z) = \overline{S}$, the first-order conditions corresponding to (26) are

$$q : \omega \cdot [P_R'(q)q + P_R(q)] - a_E \sigma^2 \omega^2 q = 0, \tag{31}$$
$$\omega : P_R(q)q - p_F^* \overline{S} - a_E \sigma^2 \omega q^2 = 0, \tag{32}$$

evaluated at $q = q^*(z)$ and $\omega = \omega^*(z)$. Solving (32) yields

$$\omega^* = \frac{P_R(q^*(z))q^*(z) - p_F^* \overline{S}}{a_E \sigma^2 q^*(z)^2}, \tag{33}$$

which does not depend on z directly. Next, plugging $S^*(z) = \overline{S}$ and (33) into the market-clearing condition $z = (1 - \omega^*(z))S^*(z)$ and solving for the inverse financial demand yields

$$D^*(z) = P_R(q^*(z))q^*(z)/\overline{S} - \left(1 - \frac{z}{\overline{S}}\right) a_E \sigma^2 q^*(z)^2/\overline{S}. \tag{34}$$

Plugging (34) into the investor's maximization problem yields

$$\max_z \left\{ W_I + \left(1 - \frac{z}{\overline{S}}\right) a_E \sigma^2 q^*(z)^2 z/\overline{S} - a_I \sigma^2 q^*(z)^2 z^2/(2\overline{S}^2) \right\} \tag{35}$$

where, from (31), $q^*(z)$ does not depend on z. The first-order condition is

$$\left(1 - \frac{2z}{\overline{S}}\right) a_E \sigma^2 q^*(z)^2/\overline{S} - a_I \sigma^2 q^*(z)^2 z/\overline{S}^2 = 0 \tag{36}$$

evaluated at $z = z^*$ yielding (29). Next, plugging (29) into (34) yields (30). Plugging (30) into (33) yields (28). □

As in Nash and Stackelberg with a leading entrepreneur, the allocation of risk under Stackelberg with a leading investor depends on the risk-aversion coefficients. However, in Stackelberg with a leading investor, it is the investor who has to take into account the effect of an increase in shares demanded in the financial price through the marginal risk cost of the entrepreneur. Hence, the fraction of shares sold under Stackelberg with a leading investor is less than under Nash, regardless of the restriction imposed on Ω_S.

Under Stackelberg, the fraction of shares sold can be smaller or bigger, depending on the relative size of a_I and a_E. Since (28) is always larger than (8), the right-hand side of (27) is also larger than in (7), so the equilibrium output under Stackelberg with a leading investor is smaller than the Nash competitive equilibrium[17].

The signs of the effects of the risk coefficients on ω and q remain unchanged. The value of $\omega^* = \frac{a_I + a_E}{a_I + 2a_E}$ is 1 when a_E tends to zero; but when a_I

[17]From expression (27), it can be shown that the equilibrium output is no longer Pareto optimal.

tends to zero ω equals $1/2$. This is quite interesting, because now, even when the investor is risk neutral, he does not buy all the shares. Indeed, acting as a leader, he prefers to buy less than half the shares to force the entrepreneur into selling with a greater discount. When a_I approaches infinity, ω tends to 1. However, when a_E approaches infinity, ω tends $1/2$, essentially for the same reason.

A corresponding behavior can be inferred about q^*. When a_E tends to zero or when a_I approaches infinity, ω^* goes to 1 and q^* solves $P_R'(q^*)q^* + P_R(q^*) = a_E\sigma^2 q^*$. When a_I tends to zero, ω^* goes to $1/2$ and q^* solves $P_R'(q^*)q^* + P_R(q^*) = a_E\sigma^2 q^*/2$. However, when a_E approaches infinity, the problem becomes more complicated: ω goes to $1/2$, but as q^* solves $P_R'(q^*)q^* + P_R(q^*) = a_E\sigma^2 q^*$, the right-hand side of the first-order condition goes to infinity, forcing q^* to tend to zero. This happens because, as the entrepreneur cannot sell all the shares, he must always support some of the risk; when his risk aversion increases, the only way to compensate is to decrease the output towards zero.

4 Final Remarks

In this paper, we have discussed the issue of integrating the real and financial markets in a simple but robust model. An equilibrium exists when the entrepreneur (the issuer of the shares to be sold) acts as a leader in the financial market; if the entrepreneur acts as a price-taker in the financial market some restrictions must be imposed on the set for the total number of shares issued.

In all situations here studied, financial access leads to better risk-sharing and increased output, but only the competitive financial market is Pareto optimal. Also, increases in uncertainty or on the risk aversion coefficients of the agents lead to a decrease in output.

The interaction between real and financial markets deserves further researching, namely introducing asymmetric information on some of the parameters, a multi-period time horizon and the possibility of learning and experimenting.

A Pareto Optimality

To see Pareto optimality, notice that (13) is the solution of

$$\max_{\omega,q} CE_E = \max_{\omega,q} \left\{\omega \cdot P_R(q)q + p_F^* \cdot (1-\omega) \cdot S - a_E\sigma^2\omega^2q^2/2\right\} \qquad (37)$$

where $S = \overline{S}$, and subject to $W^* = W_I + (P_R(q)q/S - p_F^*)z - a_I\sigma^2(q/S)^2z^2/2$, for $W^* > 0$. Hence, the Lagrangian is

$$\begin{aligned}\mathcal{L} &= \omega \cdot P_R(q)q + p_F^* \cdot (1-\omega) \cdot S - a_E\sigma^2\omega^2q^2/2 \\ &\quad + \lambda\left(W^* - W_I - (P_R(q^*)q/S + p_F^*)z + a_I\sigma^2(q/S)^2z^2/2\right), \end{aligned} \qquad (38)$$

so that

$$\frac{\partial\mathcal{L}}{\partial q} = \omega\cdot[P_R'(q^*)q^* + P_R(q^*)] - a_E\sigma^2\omega^2q + \lambda\left[(-P_R'(q^*)q^* - P_R(q^*))z/\overline{S} + a_I\sigma^2(z/\overline{S})^2q\right] = 0. \qquad (39)$$

Setting W^* so that $\lambda = -1$, and using the market-clearing condition $z = (1-\omega)\overline{S}$ or $z/\overline{S} = 1-\omega$ into (39) yields (13).

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