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Strategic Interactions in a One-Sector Growth Model

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Abstract:
We study the effect of dynamic and investment externalities in a one-sector growth model. In our model, two agents interact strategically in the utilization of capital for consumption, savings, and investment in technical progress. We consider two types of investment choices: complements and substitutes. For each case, we derive the equilibrium and provide the corresponding stationary distribution. We then compare the equilibrium with the social planner's solution.

Keywords: Capital accumulation, Dynamic game, Growth, Investment, Technical progress

JEL Classification: C72, C73, D81, D92, O40

1 Introduction

The relation between present decisions and future outcomes forms the basis for studying a dynamic economy. The one-sector, aggregate, growth model provides a simple way of addressing dynamic issues as studied in Solow (1956) for the positive non-stochastic case, Mirman (1972, 1973) for the positive stochastic case, Cass (1965) and Koopmans (1965) for the optimal non-stochastic case, and Brock and Mirman (1972) and Mirman and Zilcha (1975) for the optimal stochastic case. The optimal growth analysis has also been extended to non-classical settings (Amir et al., 1991) as well as learning environments (Koulovatianos et al., 2009). Although it is important to understand the optimal path of aggregate capital, optimal growth analysis precludes studying the strategic interactions of agents in competing for capital. Specifically, in utilizing capital, agents in the economy need to consider the interests of their competitors. This is particularly relevant for capital such as airports, harbors, roads, pipe lines, transmission grids, railroads, telecommunications lines, stocks of natural resources, and energy.

Externalities arise naturally in models with strategic interactions. Different types of strategic interactions imply different sorts of externalities. The first one to be studied in a dynamic strategic context was the *dynamic* externality (Mirman, 1979; Levhari and Mirman, 1980), i.e., the utilization of capital by one agent has an effect on the other agents' availability of capital in the future. For instance, an agent that increases usage of telecommunication lines reduces the effective use of this capital structure by other agents. The dynamic externality yields over-utilization of the capital, and, therefore, a smaller steady state. In a more recent paper, Koulovatianos and Mirman (2007) studies the link between market structure and industry dynamics. The interaction of entities in the market for the final good gives rise to a *market* externality. Koulovatianos and Mirman (2007) shows that the combination of market and dynamic externalities has an ambiguous effect on the overall utilization of the capital as well as the steady state. The study of strategic interactions and its effect on the path of capital is not limited to the one-sector growth model. In a two-sector growth model, Fischer and Mirman

(1992, 1996) consider a dynamic game of exploiting two resources with both dynamic and capital externalities.[1] Datta and Mirman (1999, 2000) study strategic interactions with dynamic and capital externalities in a two-sector growth model with trading.

In the one-sector growth model, dynamic and market externalities are not the only externalities affecting the path of capital. Various agents in the economy also invest in technical progress. The interaction of agents investing in maintaining and improving the efficiency of the stock of capital gives rise to an *investment* externality, i.e., the investment in technical progress by one agent has an effect on the other agent's payoff through the appreciation of the future stock. For instance, if one agent improves or maintains the effectiveness of telecommunication lines, then all agents benefit.

It is the purpose of this paper to study the dynamics of the economy when agents interact strategically. Specifically, in a one-sector growth model, we study the equilibrium path of capital in the presence of both dynamic and investment externalities.[2] To that end, we adapt the Levhari and Mirman (1980) framework to gain insight into the effect of strategic investment in technical progress on both behavior and the dynamics of capital. In our model, there are two agents interacting strategically. Each period, agents divide capital into consumption, savings and investment in technical progress. Consumption yields immediate payoffs. Savings and investment in technical progress have an effect on the future level of capital. Specifically, savings refers to the amount of present capital used in the production for the next period. Investment in technical progress modifies the technical possibilities in the production process. Together, savings and investment in technical progress influence the amount of capital available in the next period. Since both agents save and invest in technical progress, there are dynamic and investment externalities. That is, an agent's savings and investment decisions

[1] Capital externalities are called biological externalities in the context of natural resources.

[2] We consider the equilibrium path of capital corresponding to the Cournot equilibrium. For the study of the equilibrium path of capital in a decentralized economy, see Mirman et al. (2008).

have an effect on the other agent's future payoffs via production.[3]

We consider two types of investment choices: complements and substitutes. The investment choices are complements when the input of both agents is necessary to yield technical progress. In other words, if one agent fails to invest in technical progress, there is no production in the next period. The investment choices are substitutes if only total investment matters for the future production. That is, the individual contribution in technical progress has no effect on future capital except through the total amount. For each case, we derive the dynamic Cournot-Nash equilibrium under finite and infinite horizons. We also provide the stationary distribution corresponding to the infinite-horizon equilibrium. We then compare the outcome of the game with the social planner's solution. We show that there is a tragedy of the commons in the sense that the game (compared to social planning) yields more utilization of the stock. In addition, the game leads to an increase in consumption and a decrease in investment for technical progress. As a result, the investment externality has a negative effect on the stationary distribution of capital.

The paper is organized as follows. Section 2 presents the model and defines the equilibrium. Section 3 characterizes the equilibrium under for both complements and substitutes and provides the stationary distributions under a game. Section 4 studies the effect of the investment externality by comparing the equilibrium outcomes with the solution of the social planner. Section 5 offers some concluding remarks.

2 Model and Equilibrium

In this section, we present a dynamic game with two agents. Each period, agents divide capital into consumption, savings and investment in technical

[3]The investment externality in this paper is different from the biological (or capital) externality studied in two-sector growth models (Fischer and Mirman, 1992, 1996; Datta and Mirman, 1999, 2000). The biological externality arises directly from the presence of two goods. That is, the future stock of one type of capital depends on the interaction of the savings for both types of capital. In our one-sector growth model, there is only one stock of capital, i.e., aggregate capital.

progress. Consumption yields immediate payoffs. Savings and investment in technical progress have an effect on the future level of capital. We first present the growth model. We then define the recursive Cournot-Nash equilibrium. In the subsequent sections, we characterize the equilibrium under both complementary and substitutionary investment choices. We then compare the equilibrium outcomes with the social planner's solution.

Let y_t be the stock of capital available at the beginning of period t. Absent utilization and investment, the stock of capital evolves stochastically according to the rule[4]

$$\tilde{y}_{t+1} = f(y_t, \tilde{\alpha}_t) \tag{1}$$

where $f(\cdot)$ is the transition function and $\tilde{\alpha}_t$ is an i.i.d. random technological shock in period t, i.e., the shock is realized in period $t+1$.

In period t, for $j = 1, 2$, agent j utilizes $e_{j,t}$ units of capital in order to consume $c_{j,t}$ units and invests $i_{j,t}$ units for technical progress, i.e., $e_{j,t} = c_{j,t} + i_{j,t}$. Consumption yields immediate payoffs $\pi(c_{j,t})$. The agents' utilization of the capital and their level of investment in technical progress have an effect on the future stock. Using (1),

$$\tilde{y}_{t+1} = g(i_{1,t}, i_{2,t}, \tilde{\boldsymbol{\eta}}_t) \cdot f(y_t - e_{1,t} - e_{2,t}, \tilde{\alpha}_t) \tag{2}$$

where $g(\cdot)$ is the investment function and $\tilde{\boldsymbol{\eta}}_t$ is a vector of i.i.d. shocks in period t. To simplify notation, the t-subscript for indexing time is hereafter removed and the hat sign is used to indicate the value of a variable in the subsequent period, i.e., y is stock today and, given any realizations of $\boldsymbol{\eta}$ and α,

$$\hat{y} = g(i_1, i_2, \boldsymbol{\eta}) \cdot f(y - e_1 - e_2, \alpha) \tag{3}$$

is stock tomorrow. From (3), investment in technical progress is needed to maintain capital and ensure its future use.

To distinguish among different horizons of the dynamic game, we use the index $\tau = 0, 1, \ldots, T$. Given a horizon and the present stock of the aggregate capital, agent j maximizes the expected sum of discounted payoffs

[4] A tilde sign distinguishes a random variable from its realization.

over utilization and investment. Formally, for $j, k = 1, 2, j \neq k$, the τ-period-horizon value function of agent j is

$$v_j^\tau(y) = \max_{e_j, i_j}\{\pi(e_j - i_j) + \delta\mathbb{E}v_j^{\tau-1}(g(i_j, i_k, \tilde{\boldsymbol{\eta}}) \cdot f(y - e_j - e_k, \tilde{\alpha}))\} \tag{4}$$

where $c_j = e_j - i_j$ and $\mathbb{E}$ is the expectation operator for $\tilde{\boldsymbol{\eta}}$ and $\tilde{\alpha}$. From (4), agent k's choices have an effect on agent j's expected sum of discounted payoffs through the dynamics of capital.

In general, in a dynamic game, the value function defined in (4) might not be concave (Mirman, 1979). In addition, our model includes two inherently dynamic decisions for each agent as well as several random shocks. In order to characterize the equilibrium and study its properties under different cases of investment choices, we resort to a modified version of the Levhari and Mirman (1980) framework. The following assumptions hold for the remainder of the paper. We leave the investment function unspecified for the moment and consider several types of investment choices in the next sections.

Assumption 2.1. *The joint p.d.f. of $\tilde{\boldsymbol{\eta}}$ and $\tilde{\alpha}$ is $\phi(\boldsymbol{\eta}, \alpha), \boldsymbol{\eta} \in (0,1)^2, \alpha \in (0,1)$. Let $\overline{\boldsymbol{\eta}} \equiv \mathbb{E}\tilde{\boldsymbol{\eta}}$ and $\overline{\alpha} \equiv \mathbb{E}\tilde{\alpha}$ be the means of the random shocks.*

Assumption 2.2. *For $j = 1, 2$, $\pi(c_j) = \ln c_j$.*

Assumption 2.3. *For $\alpha \in (0,1)$, $f(y - e_j - e_k, \alpha) = (y - e_j - e_k)^\alpha$.*

We now define the recursive Cournot-Nash equilibrium for a T-period-horizon game (Levhari and Mirman, 1980). The equilibrium consists of the strategies of the two agents for every horizon from the first period (when there are T periods left) to the last period (when there is no horizon). Without loss of generality, we assume that in the last period the two agents split the stock equally and do not invest. The assumption of a log utility function implies that the allocation of the stock in the last period has no effect on the dynamic game. Condition 1 states the behavior in the last period, i.e., when the horizon is $\tau = 0$. Condition 2 states the recursive equilibrium for every horizon of the game. Expression (6) for $\tau = 1$ is consistent with statement 1, i.e., for all j, $V_j^0(y) = \ln(E_j^0(y) - I_j^0(y))$. Expression (6) for

$\tau = 2, \ldots, T-1$ reflects the recursive nature of the equilibrium in which the equilibrium continuation value function for a τ-period horizon depends on the equilibrium strategies for τ'-period horizons, $\tau > \tau' \geq 0$.

Definition 2.4. *The tuple $\{E_1^\tau(y), I_1^\tau(y), E_2^\tau(y), I_2^\tau(y)\}_{\tau=0}^T$ is a recursive Cournot-Nash equilibrium for a T-period-horizon game if, for all $y > 0$,*

1. *For $\tau = 0$, $E_1^0(y) = E_2^0(y) = y/2, I_1^0(y) = I_2^0(y) = 0$.*

2. *For $\tau = 1, 2, \ldots, T$, for $j, k = 1, 2, j \neq k$, given $\{E_k^\tau(y), I_k^\tau(y)\}$ and $\{E_1^t(y), I_1^t(y), E_2^t(y), I_2^t(y)\}_{t=0}^{\tau-1}$,*

$$\{E_j^\tau(y), I_j^\tau(y)\} = \arg\max_{e_j, i_j} \Bigg\{ \ln(e_j - i_j) + \delta \int V_j^{\tau-1}(g(i_j, I_k^\tau(y), \boldsymbol{\eta}) \cdot (y - e_j - E_k^\tau(y))^\alpha) \cdot \phi(\boldsymbol{\eta}, \alpha) \mathrm{d}\boldsymbol{\eta} \mathrm{d}\alpha \Bigg\} \tag{5}$$

where, for any $x > 0$,

$$V_j^{\tau-1}(x) = \begin{cases} \ln(x/2), & \tau = 1 \\ \ln(E_j^{\tau-1}(x) - I_j^{\tau-1}(x)) + \delta \int V_j^{\tau-2}(\Lambda) \cdot \phi(\boldsymbol{\eta}, \alpha) \mathrm{d}\boldsymbol{\eta} \mathrm{d}\alpha, & \tau = 2, 3, ..., T \end{cases} \tag{6}$$

with

$$\Lambda \equiv g(I_1^{\tau-1}(x), I_2^{\tau-1}(x), \boldsymbol{\eta}) \cdot \left(x - \sum_{s=1}^{2} E_s^{\tau-1}(x)\right)^\alpha. \tag{7}$$

3 Equilibrium Characterization

In this section, we fully characterize the equilibrium for any finite horizon. We then show that the limit of the finite-horizon equilibrium exists. In other words, there exists an equilibrium for the infinite horizon that is consistent with the sequence of finite-horizon equilibrium. We then provide the stationary distribution for capital corresponding to the limiting case.

We begin with the case of complementary investment choices. We then repeat the analysis for the case of substitutionary investment choices. The main difference between complements and substitutes concern uniqueness. When investment choices are complementary, the equilibrium is unique whereas there is a continuum of equilibrium points with substitutionary investment choices. However, regardless of the type of investment, the stationary distribution of capital is unique. In the next section, we compare the equilibrium with the solution of the social planner.

3.1 Complementary Investment Choices

When investment choices are complementary, the investment function is specified as

$$g(i_1, i_2, \boldsymbol{\eta}) = i_1^{\eta_1} i_2^{\eta_2}, \tag{8}$$

$\boldsymbol{\eta} \equiv [\eta_1, \eta_2]$. Using (8), (3) is rewritten as

$$\hat{y} = i_1^{\eta_1} i_2^{\eta_2} (y - e_1 - e_2)^{\alpha}. \tag{9}$$

The investment term $i_1^{\eta_1} i_2^{\eta_2}$ reflects the complementarity of the agents' investments. The shocks η_1 and η_2 measure the individual contribution of the level of investment toward the future stock.

Proposition 3.1 provides the utilization level as well as the levels for consumption and investment corresponding to the unique equilibrium for any finite horizon. The equilibrium displays certainty equivalence, i.e., the means of the shocks are the only moments of the distribution to have an effect on decisions. Moreover, the equilibrium is in general asymmetric unless the means of the investment shocks are identical.

Proposition 3.1. *Suppose that the investment choices are complementary. Then, there exists a unique recursive Cournot-Nash equilibrium for a T-period game, $T = 1, 2, \ldots$ In equilibrium, for $\tau = 0, 1, \ldots, T$, for $j = 1, 2$, agent j utilizes*

$$E_j^{\tau}(y) = \frac{1 + \delta \overline{\eta}_j \left(\sum_{t=0}^{\tau-1} \delta^t (\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha})^t \right)}{2 + \delta(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha}) \left(\sum_{t=0}^{\tau-1} \delta^t (\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha})^t \right)} y \tag{10}$$

units of capital for the production of

$$C_j^\tau(y) = \frac{1}{2+\delta(\overline{\eta}_1+\overline{\eta}_2+\overline{\alpha})\left(\sum_{t=0}^{\tau-1}\delta^t(\overline{\eta}_1+\overline{\eta}_2+\overline{\alpha})^t\right)}y \tag{11}$$

units of consumption and

$$I_j^\tau(y) = \frac{\delta\overline{\eta}_j\left(\sum_{t=0}^{\tau-1}\delta^t(\overline{\eta}_1+\overline{\eta}_2+\overline{\alpha})^t\right)}{2+\delta(\overline{\eta}_1+\overline{\eta}_2+\overline{\alpha})\left(\sum_{t=0}^{\tau-1}\delta^t\left(\overline{\eta}_1+\overline{\eta}_2+\overline{\alpha}\right)^t\right)}y \tag{12}$$

units of investment.

Proof. We first derive utilization, investment, and value functions in the one-period horizon. We then consider a τ-period horizon and solve for utilization, investment and value functions recursively. We finally impose the initial condition given by the one-period-horizon solution.

1. Consider first the one-period horizon. Using (5), (6), and (9), for $j,k = 1,2, j\neq k$, given $\{E_k^1(y), I_k^1(y)\}$, agent j's one-period-horizon optimal policies satisfy

$$\begin{aligned}\{E_j^1(y), I_j^1(y)\} = &\arg\max_{e_j,i_j}\big\{\ln(e_j-i_j)+\delta\overline{\eta}_j\ln i_j+\delta\overline{\eta}_k\ln I_k^1(y)\\ &+\delta\overline{\alpha}\ln(y-e_j-E_k^1(y))-\delta\ln 2\big\}.\end{aligned} \tag{13}$$

   The first-order conditions corresponding to (13) are

$$e_j : \frac{1}{e_j-i_j} = \frac{\delta\overline{\alpha}}{y-e_j-E_k^1(y)}, \tag{14}$$

$$i_j : \frac{1}{e_j-i_j} = \frac{\delta\overline{\eta}_j}{i_j}, \tag{15}$$

   evaluated at $e_j = E_j^1(y)$ and $i_j = I_j^1(y)$. Since the Hessian matrix is negative definite, the second-order condition holds. For $j,k = 1,2, j\neq k$, solving (14) and (15) for the equilibrium yields the unique solution for one-period-horizon utilization and investment,

$$E_j^1(y) = \frac{1+\delta\overline{\eta}_j}{2+\delta\left(\overline{\eta}_1+\overline{\eta}_2+\overline{\alpha}\right)}y, \tag{16}$$

$$I_j^1(y) = \frac{\delta\overline{\eta}_j}{2+\delta\left(\overline{\eta}_1+\overline{\eta}_2+\overline{\alpha}\right)}y. \tag{17}$$

Plugging (16) and (17) for the two agents into the objective function in (13) yields

$$V_j^1(y) = (1+\delta(\overline{\eta}_1+\overline{\eta}_2+\overline{\alpha}))\ln y + \Psi_1, \tag{18}$$

where Ψ_1 is a constant for the one-period horizon that has no effect on the solution.

2. Having solved for the one-period horizon, we consider next a τ-period horizon for which the continuation value function is of the form $V_j^{\tau-1}(y) = \kappa_{\tau-1}\ln y + \Psi_{\tau-1}$ where $\kappa_{\tau-1}$ and $\Psi_{\tau-1}$ are constants. For $j,k=1,2, j\neq k$, given $V_j^{\tau-1}(y) = \kappa_{\tau-1}\ln y + \Psi_{\tau-1}$ and $\{E_k^\tau(y), I_k^\tau(y)\}$, agent j's τ-period-horizon optimal policies satisfy

$$\begin{aligned}\{E_j^\tau(y), I_j^\tau(y)\} = &\arg\max_{e_j,i_j}\left\{\ln(e_j-i_j)+\delta\overline{\eta}_j\kappa_{\tau-1}\ln i_j+\delta\overline{\eta}_k\kappa_{\tau-1}\ln I_k^\tau(y)\right.\\ &\left.+\delta\overline{\alpha}\kappa_{\tau-1}\ln(y-e_j-E_k^\tau(y))+\delta\Psi_{\tau-1}\right\}.\end{aligned} \tag{19}$$

The first-order conditions corresponding to (19) are

$$e_j : \frac{1}{e_j-i_j} = \frac{\delta\overline{\alpha}\kappa_{\tau-1}}{y-e_j-E_k^\tau(y)} \tag{20}$$

$$i_j : \frac{1}{e_j-i_j} = \frac{\delta\overline{\eta}_j\kappa_{\tau-1}}{i_j} \tag{21}$$

evaluated at $e_j = E_j^\tau(y)$ and $i_j = I_j^\tau(y)$. Since the Hessian matrix is negative definite, the second-order condition holds. For $j,k=1,2, j\neq k$, solving (20) and (21) for the equilibrium yields the unique solution for τ-period utilization and investment,

$$E_j^\tau(y) = \frac{1 + \delta\overline{\eta}_j\kappa_{\tau-1}}{2 + \delta\kappa_{\tau-1}(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha})}y \tag{22}$$

$$I_j^\tau(y) = \frac{\delta\overline{\eta}_j\kappa_{\tau-1}}{2 + \delta\kappa_{\tau-1}(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha})}y. \tag{23}$$

Plugging (22) and (23) for the two agents into the objective function in (19) yields

$$V_j^\tau(y) = (1 + \delta\kappa_{\tau-1}(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha}))\ln y + \Delta_\tau, \tag{24}$$

$$\equiv \kappa_\tau \ln y + \Psi_\tau, \tag{25}$$

where Δ_τ and Ψ_τ are constants that we ignore since they have no effect on the solution. Hence,

$$\kappa_\tau = 1 + \delta\kappa_{\tau-1}(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha}) \tag{26}$$

with, from (18), initial condition

$$\kappa_1 = 1 + \delta(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha}). \tag{27}$$

From (26) and (27), it follows that

$$\kappa_\tau = \sum_{t=0}^{\tau} \delta^t(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha})^t. \tag{28}$$

Plugging (28) into (22) and (23) yields (10) and (12), respectively. Plugging (10) and (12) into $C_j^\tau(y) = E_j^\tau(y) - I_j^\tau(y)$ yields (11).

□

We now show that there is no disparity between the finite and infinite horizons. Specifically, using Proposition 3.1, we show that the limits of the equilibrium outcomes exist. In other words, there exists a unique equilibrium for the infinite horizon that is consistent with the sequence of finite-horizon equilibrium. We then use these limiting outcomes to derive the unique stationary distribution for capital. Proposition 3.2 provides the limits of the equilibrium outcomes.

Proposition 3.2. *Suppose that the investment choices are complementary. If* $\overline{\eta}_1 + \overline{\eta}_2 + \alpha \in (0,1)$*, then for* $j,k = 1,2, j \neq k$*,* $\lim_{T\to\infty} E_j^T(y) \equiv E_j^\infty(y)$*,* $\lim_{T\to\infty} C_j^T(y) \equiv C_j^\infty(y)$*, and* $\lim_{T\to\infty} I_j^T(y) \equiv I_j^\infty(y)$ *exist such that*

$$E_j^\infty(y) = \frac{1-\delta(\overline{\eta}_k + \overline{\alpha})}{2-\delta(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha})} y, \tag{29}$$

$$C_j^\infty(y) = \frac{1-\delta(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha})}{2-\delta(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha})} y, \tag{30}$$

$$I_j^\infty(y) = \frac{\delta\overline{\eta}_j}{2-\delta(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha})} y. \tag{31}$$

Proof. Given that $\overline{\eta}_1 + \overline{\eta}_2 + \alpha \in (0,1)$, taking limits of (10), (11) and (12) yields (29), (30) and (31), respectively. □

Using the limiting outcomes, Proposition 3.3 provides the stationary distribution of capital. Due to the fact that the equilibrium displays certainty equivalence, the stationary distribution depends directly on the means of the shocks. However, through (9), the stationary distribution of capital depends on the distribution of the shocks, i.e., first and higher moments.

Proposition 3.3. *Suppose that the investment choices are complementary. Then, the stationary distribution of capital is defined by*

$$\tilde{Y} = \left(\frac{\overline{\eta}_1^{\tilde{\eta}_1} \overline{\eta}_2^{\tilde{\eta}_2} \overline{\alpha}^{\tilde{\alpha}} \delta^{\tilde{\eta}_1+\tilde{\eta}_2+\tilde{\alpha}}}{\left(2-\delta\left(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha}\right)\right)^{\tilde{\eta}_1+\tilde{\eta}_2+\tilde{\alpha}}} \right)^{\frac{1}{1-(\tilde{\eta}_1+\tilde{\eta}_2+\tilde{\alpha})}}. \tag{32}$$

Proof. Plugging (29) and (31) into (9) and solving for $\tilde{Y} = \hat{y} = y$ yields (32). □

3.2 Substitutionary Investment Choices

When investment choices are substitutionary, the investment function is specified as

$$\phi(i_1, i_2, \boldsymbol{\eta}) = (i_1 + i_2)^{\eta}, \tag{33}$$

$\boldsymbol{\eta} \equiv \eta$. Using (33), (3) is rewritten as

$$\hat{y} = (i_1 + i_2)^{\eta}(y - e_1 - e_2)^{\alpha}, \tag{34}$$

The investment term $(i_1 + i_2)^{\eta}$ reflects the perfect substitutability of the agents' investments.

Unlike the case of complementary investment choices, the equilibrium is not unique. In fact, when investment choices are substitutionary, there is a continuum of equilibrium that admits any allocation of the total investment between the two agents but leaves total investment unchanged. The multiplicity of the equilibrium has no bearing on the dynamics of the capital and thus on agents' future payoffs since, from (34), only total investment matters.

Proposition 3.4 states the properties of the equilibrium. The multiplicity of the equilibrium is reflected by the allocation of the investment between agents 1 and 2. That is, for $j = 1, 2$, $\gamma_{1,\tau} \in [0, 1]$ is the fraction of total investment undertaken by agent j when the horizon is τ periods. Hence, $\gamma_{1,\tau} + \gamma_{2,\tau} = 1$.

Proposition 3.4. *Suppose that the investment choices are substitutionary. Then, there exists a continuum of recursive Cournot-Nash equilibrium for a T-period game, $T = 1, 2, \ldots$ For any equilibrium, for $\tau = 1, ..., T$,*

1. $C_1^{\tau}(y) = C_2^{\tau}(y)$.
2. *For $j = 1, 2$ and for any allocation $\{\gamma_{1,\tau}, \gamma_{2,\tau}\}$ such that $\gamma_{1,\tau}, \gamma_{2,\tau} \in [0, 1], \gamma_{1,\tau} + \gamma_{2,\tau} = 1$, $I_j^{\tau}(y) = \gamma_{j,\tau} \cdot (I_1^{\tau}(y) + I_2^{\tau}(y))$.*

Proof. See the proof of Proposition 3.5. □

Proposition 3.5 provides the utilization level as well as the production levels for consumption and investment choices corresponding to the equilibrium for any finite horizon. As in the case of complementary investment choices, the equilibrium displays certainty equivalence.

Proposition 3.5. *Suppose that the investment choices are substitutionary. Then, in equilibrium, for* $\tau = 0, 1, \ldots, T$*, for* $j = 1, 2$*, given an allocation* $\{\gamma_{1,\tau}, \gamma_{2,\tau}\}$*, agent* j *utilizes*

$$E_j^\tau(y) = \frac{1 + \gamma_{j,\tau}\delta\overline{\eta}\left(\sum_{t=0}^{\tau-1} \delta^t \left(\overline{\eta} + \overline{\alpha}\right)^t\right)}{2 + \delta(\overline{\eta} + \overline{\alpha})\left(\sum_{t=0}^{\tau-1} \delta^t \left(\overline{\eta} + \overline{\alpha}\right)^t\right)} y \tag{35}$$

units of capital for the production of

$$C_j^\tau(y) = \frac{1}{2 + \delta(\overline{\eta} + \overline{\alpha})\left(\sum_{t=0}^{\tau-1} \delta^t \left(\overline{\eta} + \overline{\alpha}\right)^t\right)} y \tag{36}$$

units of consumption and

$$I_j^\tau(y) = \frac{\gamma_{j,\tau}\delta\overline{\eta}\left(\sum_{t=0}^{\tau-1} \delta^t \left(\overline{\eta} + \overline{\alpha}\right)^t\right)}{2 + \delta(\overline{\eta} + \overline{\alpha})\left(\sum_{t=0}^{\tau-1} \delta^t (\overline{\eta} + \overline{\alpha})^t\right)} y \tag{37}$$

units of investment.

Proof. We first derive utilization, investment, and value functions in the one-period horizon. We then consider a τ-period horizon and solve for utilization, investment and value functions recursively. We finally impose the initial condition given by the one-period-horizon solution.

1. Consider first the one-period horizon. Using (5), (6) and (34), for $j, k = 1, 2, j \neq k$, given $\{E_k^1(y), I_k^1(y)\}$, agent j's one-period-horizon optimal policies satisfy

$$\begin{aligned}\{E_j^1(y), I_j^1(y)\} = &\arg\max_{e_j, i_j} \big\{\ln(e_j - i_j) + \delta\overline{\eta}\ln(i_j + I_k^1(y)) \\ &+\delta\overline{\alpha}\ln(y - e_j - E_k^1(y)) - \delta\ln 2\big\}.\end{aligned} \tag{38}$$

The first-order conditions corresponding to (38) are

$$e_j : \frac{1}{e_j - i_j} = \frac{\delta\overline{\alpha}}{y - e_j - E_k^1(y)}, \tag{39}$$

$$i_j : \frac{1}{e_j - i_j} = \frac{\delta\overline{\eta}}{i_j + I_k^1(y)}, \tag{40}$$

evaluated at $e_j = E_j^1(y)$ and $i_j = I_j^1(y)$. Since the Hessian matrix is negative definite, the second-order condition holds. However, individual investment cannot be determined because $I_j^1(y)$ and $I_k^1(y)$ have an effect on equilibrium condition only through their sum. To see this, for $j = 1, 2$, plugging $C_j^1(y) = E_j^1(y) - I_j^1(y)$ into (39) and (40) and rearranging yields the system

$$\frac{1}{C_1^1(y)} = \frac{\delta\overline{\alpha}}{y - C_1^1(y) - I_1^1(y) - C_2^1(y) - I_2^1(y)}, \tag{41}$$

$$\frac{\delta\overline{\eta}}{I_1^1(y) + I_2^1(y)} = \frac{\delta\overline{\alpha}}{y - C_1^1(y) - I_1^1(y) - C_2^1(y) - I_2^1(y)}, \tag{42}$$

$$\frac{1}{C_2^1(y)} = \frac{\delta\overline{\eta}}{y - C_2^1(y) - I_2^1(y) - C_1^1(y) - I_1^1(y)}, \tag{43}$$

$$\frac{\delta\overline{\eta}}{I_1^1(y) + I_2^1(y)} = \frac{\delta\overline{\alpha}}{y - C_2^1(y) - I_2^1(y) - C_1^1(y) - I_1^1(y)}, \tag{44}$$

which defines the one-period-horizon solution for the equilibrium, i.e., $\{C_j^1(y), I_j^1(y)\}_{j=1}^2$. From (42) and (44), one equation is redundant, which implies that there are three equations for four unknowns. In fact, $C_j^1(y)$, $C_k^1(y)$ and $I_k^1(y) + I_j^1(y)$ have unique solutions, but $I_j^1(y)$ and $I_k^1(y)$ cannot be determined separately.

Letting $\gamma_{j,1} \in (0, 1)$ be the fraction of total investment by agent j in the one-period horizon, solving (39) and (40) for the equilibrium yields the solution for one-period-horizon utilization and investment:

$$E_j^1(y) = \frac{1 + \gamma_{j,1}\delta\overline{\eta}}{2 + \delta\left(\overline{\eta} + \overline{\alpha}\right)} y \tag{45}$$

$$I_j^1(y) = \gamma_{j,1} \frac{\delta\overline{\eta}}{2 + \delta\left(\overline{\eta} + \overline{\alpha}\right)} y. \tag{46}$$

Plugging (45) and (46) for the two agents into the objective function in (38) yields

$$V_j^1(y) = (1 + \delta(\overline{\eta} + \overline{\alpha})) \ln y + \Psi_1 \tag{47}$$

where Ψ_1 is a constant for the one-period horizon that has no effect on the solution.

2. Having solved for the one-period-horizon, we consider next a τ-period-horizon for which the continuation value function is of the form $V^{\tau-1}(y) = \kappa_{\tau-1} \ln y + \Psi_{\tau-1}$ where $\kappa_{\tau-1}$ and $\Psi_{\tau-1}$ are unknown constants. For $j, k = 1, 2, j \neq k$, given $V^{\tau-1}(y) = \kappa_{\tau-1} \ln y + \Psi_{\tau-1}$ and $\{E_k^\tau(y), I_k^\tau(y)\}$, agent j's τ-period-horizon optimal policies satisfy

$$\begin{aligned}\{E_j^\tau(y), I_j^\tau(y)\} = \arg\max_{e_j, i_j} \{&\ln(e_j - i_j) + \delta\overline{\eta}\kappa_{\tau-1} \ln(i_j + I_k^\tau(y)) \\ &+\delta\overline{\alpha}\kappa_{\tau-1} \ln(y - e_j - E_k^\tau(y)) + \delta\Psi_{\tau-1}\}.\end{aligned} \tag{48}$$

The first-order conditions corresponding to (48) are

$$e_j : \frac{1}{e_j - i_j} = \frac{\delta\overline{\alpha}\kappa_{\tau-1}}{y - e_j - E_k^\tau(y)}, \tag{49}$$

$$i_j : \frac{1}{e_j - i_j} = \frac{\delta\overline{\eta}\kappa_{\tau-1}}{i_j + I_k^\tau(y)} \tag{50}$$

evaluated at $e_j = E_j^\tau(y)$ and $i_j = I_j^\tau(y)$. Since the Hessian matrix is negative definite, the second-order condition holds. However, as noted in the one-period-horizon, individual investment cannot be determined because $I_j^1(y)$ and $I_k^1(y)$ have an effect on equilibrium condition only through their sum. Letting $\gamma_{j,\tau} \in (0, 1)$ be the fraction of total investment produced by agent j in the τ-period horizon, Solving (49) and (50) for the equilibrium yields the solution for utilization and investment,

$$E_j^\tau(y) = \frac{1 + \gamma_{\tau,j}\delta\overline{\eta}\kappa_{\tau-1}}{2 + \delta\kappa_{\tau-1}(\overline{\eta} + \overline{\alpha})} y \tag{51}$$

$$I_j^\tau(y) = \frac{\gamma_{\tau,j}\delta\overline{\eta}\kappa_{\tau-1}}{2 + \delta\kappa_{\tau-1}(\overline{\eta} + \overline{\alpha})} y. \tag{52}$$

Plugging (51) and (52) for the two agents into the objective function in (48) yields

$$V_j^\tau(y) = (1 + \delta\kappa_{\tau-1}\,(\overline{\eta} + \overline{\alpha}))\ln y + \Delta_\tau \tag{53}$$

$$\equiv \kappa_\tau \ln y + \Psi_\tau, \tag{54}$$

where Δ_τ and Ψ_τ are constants that we ignore since they have no effect on the solution. Hence,

$$\kappa_\tau = 1 + \delta\kappa_{\tau-1}\,(\overline{\eta} + \overline{\alpha}) \tag{55}$$

with, from (47), initial condition

$$\kappa_1 = 1 + \delta\,(\overline{\eta} + \overline{\alpha})\,. \tag{56}$$

From (55) and (56), it follows that

$$\kappa_\tau = \sum_{t=0}^{\tau} \delta^t (\overline{\eta} + \overline{\alpha})^t. \tag{57}$$

Plugging (57) into (51) and (52) yields (35) and (37). Plugging (35) and (37) into $C_j^\tau(y) = E_j^\tau(y) - I_j^\tau(y)$ yields (36).

□

For each point in the continuum of finite-horizon equilibrium, the limits to the finite-horizon equilibrium outcomes exist. As in the case of complementary investment choices, the case of substitutionary investment choices yields no disparity between the finite and infinite horizons. Proposition 3.6 provides the equilibrium for an infinite horizon, i.e., the limits of the equilibrium outcomes in Proposition 3.5.

Proposition 3.6. *Suppose that the investment choices are substitutionary. If* $\eta + \alpha \in (0,1)$*, then for* $j = 1, 2$*,* $\lim_{T\to\infty} E_j^T(y) \equiv E_j^\infty(y)$*,* $\lim_{T\to\infty} C_j^T(y) \equiv$

$C_j^\infty(y)$, *and* $\lim_{T\to\infty} I_j^T(y) \equiv I_j^\infty(y)$ *exist such that, given an allocation* $\{\gamma_{1,\infty}, \gamma_{2,\infty}\}$,

$$E_j^\infty(y) = \frac{1-\delta\left((1-\gamma_{j,\infty})\overline{\eta}+\overline{\alpha}\right)}{2-\delta\left(\overline{\eta}+\overline{\alpha}\right)}y \tag{58}$$

$$C_j^\infty(y) = \frac{1-\delta\left(\overline{\eta}+\overline{\alpha}\right)}{2-\delta\left(\overline{\eta}+\overline{\alpha}\right)}y \tag{59}$$

$$I_j^\infty(y) = \frac{\gamma_{j,\infty}\delta\overline{\eta}}{2-\delta\left(\overline{\eta}+\overline{\alpha}\right)}y. \tag{60}$$

Proof. Given that $\eta+\alpha \in (0,1)$, taking limits of (35), (36), and (37) yields (58), (59) and (60). □

Although the equilibrium is a continuum, the perfect substitutability of the investment implies a unique stationary distribution of capital. Hence,

Proposition 3.7. *Suppose that the investment choices are substitutionary. Then, the stationary distribution of capital is defined by*

$$\tilde{Y} = \left(\frac{\delta^{\tilde{\eta}+\tilde{\alpha}}\overline{\eta}^{\tilde{\eta}}\overline{\alpha}^{\tilde{\alpha}}}{\left(2-\delta\left(\overline{\eta}+\overline{\alpha}\right)\right)^{\tilde{\eta}+\tilde{\alpha}}}\right)^{\frac{1}{1-(\tilde{\eta}+\tilde{\alpha})}}. \tag{61}$$

Proof. Plugging (58) and (60) into (34) and solving for $\tilde{Y}=\hat{y}=y$ yields (61). □

Before proceeding with the comparison between the Cournot-Nash equilibrium and the solution of the social planner, we compare differences between complements and substitutes. Apart from the uniqueness property, by comparing Propositions 3.1 and 3.2 with Propositions 3.5 and 3.6, the policy functions for the agents' behavior are of similar form. Proposition 3.8 shows that the ordering of individual utilization, consumption, and investment depends on the means of the investment shocks.

Proposition 3.8. *Suppose that* $\overline{\eta} > (<)(=)\overline{\eta}_1+\overline{\eta}_2$*, then, under complements, extraction and consumption are highest (lowest) (equal) and investment is lowest (highest) (equal).*

Proof. Comparing (29), (30), and (31) with (58), (59) and (60) yields the result. □

4 Investment Externality

Having characterized the recursive Cournot-Nash equilibrium, we study the effect of the investment externality (combined with the dynamic externality) on behavior and the stationary distribution of capital. To that end, we first provide the social planner's solution in the infinite horizon case.[5] We then characterize the tragedy of the commons. We finally derive the stationary distributions of capital corresponding to the social planner's solution and compare them with the stationary distributions corresponding to the recursive Cournot-Nash equilibrium.

Under social planning, the infinite-horizon value function of the social planner satisfies

$$W^{\infty}(y) = \max_{\{e_j, i_j\}_{j=1}^{2}} \left\{ \ln(e_1 - i_1) + \ln(e_2 - i_2) + \delta \mathbb{E} W^{\infty}(g(i_1, i_2, \tilde{\eta}) \cdot (y - e_1 - e_2)^{\tilde{\alpha}}) \right\}, \tag{62}$$

where $g(i_1, i_2, \tilde{\eta}) = i_1^{\tilde{\eta}_1} i_2^{\tilde{\eta}_2}$ if investment choices are complements and $g(i_1, i_2, \tilde{\eta}) = (i_1 + i_2)^{\tilde{\eta}}$ if substitutes. For $j = 1, 2$, let $E_j^{*\infty}(y)$, $C_j^{*\infty}(y)$, and $I_j^{*\infty}(y)$ be the optimal solutions for utilization, consumption and investment where the symbol $*$ distinguishes optimal behavior from behavior in the Cournot-Nash equilibrium.

Proposition 4.1 provides the social planner's solution for total utilization, and total consumption and total investment under complementary and substitutionary investment choices. The introduction of the game has an effect on the comparative analysis. Under social planning with either complementary or substitutionary investment choices, there is *separation* in the sense that the investment shock $\boldsymbol{\eta}$ has no effect on total utilization whereas the shock α has no effect on total investment. However, with a game, from Propositions 3.2 and 3.6, an increase in any of the means of the investment shocks decreases total utilization and an increase in $\overline{\alpha}$ causes total investment to increase.

Proposition 4.1. *There exists a unique optimal solution to* (62).

[5] To simplify the discussion, we omit the finite-horizon case. Our results on the tragedy of the commons hold for any finite horizon.

1. *Suppose that investment choices are complementary. If* $\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha} \in (0, 1)$, *then*

$$E_1^{*\infty}(y) + E_2^{*\infty}(y) = (1 - \delta\overline{\alpha})y, \tag{63}$$

$$C_1^{*\infty}(y) + C_2^{*\infty}(y) = (1 - \delta(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha}))y, \tag{64}$$

$$I_1^{*\infty}(y) + I_2^{*\infty}(y) = \delta(\overline{\eta}_1 + \overline{\eta}_2)y. \tag{65}$$

2. *Suppose that investment choices are substitutionary. If* $\overline{\eta} + \overline{\alpha} \in (0, 1)$, *then*

$$E_1^{*\infty}(y) + E_2^{*\infty}(y) = (1 - \delta\overline{\alpha})y, \tag{66}$$

$$C_1^{*\infty}(y) + C_2^{*\infty}(y) = (1 - \delta(\overline{\eta} + \overline{\alpha}))y, \tag{67}$$

$$I_1^{*\infty}(y) + I_2^{*\infty}(y) = \delta\overline{\eta}y. \tag{68}$$

Proof. See Appendix A. □

Whether the investment choices are complementary or substitutionary, the investment externality yields a tragedy in the commons in the following sense. Under a game, total utilization increases. Moreover, consumption increases at the expense of investment.

Proposition 4.2. *Suppose that investment choices are either complementary or substitutionary. Then,*

$$E_1^{*\infty}(y) + E_2^{*\infty}(y) < E_1^{\infty}(y) + E_2^{\infty}(y), \tag{69}$$

and

$$C_1^{*\infty}(y) + C_2^{*\infty}(y) < C_1^{\infty}(y) + C_2^{\infty}(y), \tag{70}$$

$$I_1^{*\infty}(y) + I_2^{*\infty}(y) > I_1^{\infty}(y) + I_2^{\infty}(y). \tag{71}$$

Proof. Comparing Propositions 3.2, 3.6, and 4.1 yields inequalities (69), (70), and (71). □

The investment externality has an effect on the stationary distribution of capital as well. Proposition 4.3 provides the stationary distribution of capital under social planning.

Proposition 4.3. *Under social planning, the stationary distribution of capital is unique.*

1. *If investment choices are complementary, then*

$$Y^* = \left(\overline{\eta}_1^{\tilde{\eta}_1}\overline{\eta}_2^{\tilde{\eta}_2}\overline{\alpha}^{\tilde{\alpha}}\delta^{\tilde{\eta}_1+\tilde{\eta}_2+\tilde{\alpha}}\right)^{\frac{1}{1-(\tilde{\eta}_1+\tilde{\eta}_2+\tilde{\alpha})}}. \tag{72}$$

2. *If investment choices are substitutionary, then*

$$Y^* = \left(\overline{\eta}^{\tilde{\eta}}\overline{\alpha}^{\tilde{\alpha}}\delta^{\tilde{\eta}+\tilde{\alpha}}\right)^{\frac{1}{1-(\tilde{\eta}+\tilde{\alpha})}}. \tag{73}$$

Proof. If investment choices are complementary, then plugging (63), (81), and (82) into (9) and solving for $\tilde{Y}^* = \hat{y} = y$ yields (72). Next, if investment choices are substitutionary, then plugging (66) and (68) into (34) and solving for $\tilde{Y}^* = \hat{y} = y$ yields (73). □

Regardless of the type of investment choices, the effect of the investment externality on the stationary distribution is illustrated in Figure 1.[6] The two solid concave lines depict expression (2) evaluated at the highest and lowest value of the realizations of the random shocks under social planning, i.e.,

$$y_{t+1} = g(I_1^{*\infty}(y_t), I_2^{*\infty}(y_t), \boldsymbol{\eta}) \cdot f(y_t - E_1^{*\infty}(y_t) - E_2^{*\infty}(y_t), \alpha). \tag{74}$$

The two dotted concave lines also depict expression (2) evaluated at the highest and lowest value of the realizations of the random shocks but under a game, i.e.,

$$y_{t+1} = g(I_1^{\infty}(y_t), I_2^{\infty}(y_t), \boldsymbol{\eta}) \cdot f(y_t - E_1^{\infty}(y_t) - E_2^{\infty}(y_t), \alpha). \tag{75}$$

[6] When investment choices are complementary, compare (32) and (72). With substitutes, compare (61) and (73).

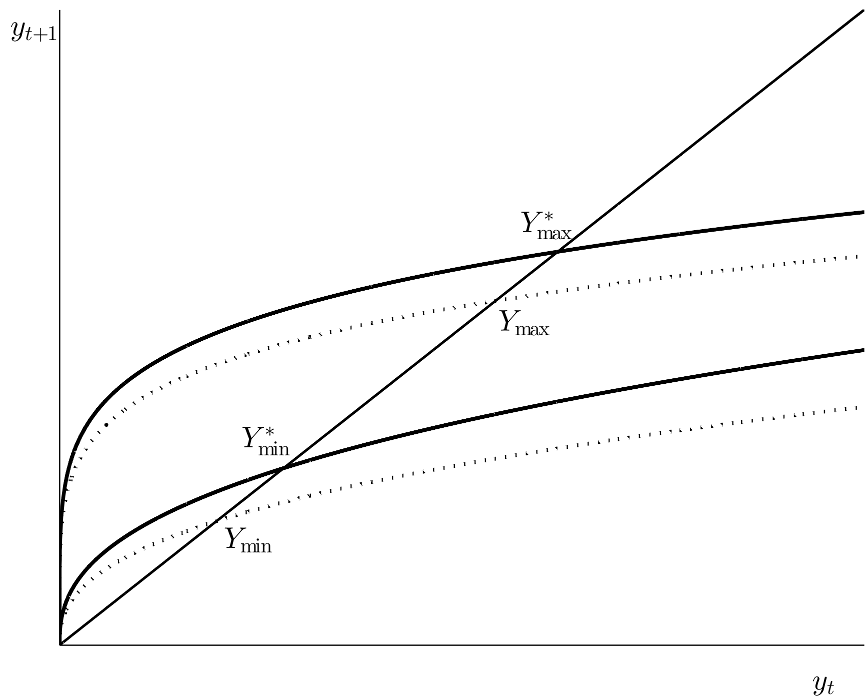

Figure 1: The Effect of the Investment Externality on the Stationary Distribution

The intersection of these lines with the 45 degree line defines the end-points of the stationary distributions under social planning and under a game.[7] Specifically, the stationary distribution under social planning has support $[Y^*_{\min}, Y^*_{\max}]$ whereas the stationary distribution under a game has support $[Y_{\min}, Y_{\max}]$. Since $Y_{\min} < Y^*_{\min}$ and $Y_{\max} < Y^*_{\max}$, the effect of the game with an investment externality is to reduce the effectiveness of the stock of aggregate capital. However, it is ambiguous whether the negative effect is strongest with complements or substitutes, i.e., it depends on the values of the parameters.

[7]Recall that in our model investment is required to maintain the capital. Without investment, the stationary distribution is degenerate at zero.

5 Final Remarks

In order to study the effect of the investment externality on utilization, production and the dynamic path of aggregate capital, we have considered a stochastic environment in which agents know the true distribution of the random shocks. However, agents generally face more than just uncertainty in outcomes since the true distributions of these shocks are never known exactly. In other words, agents generally face *structural uncertainty* because they do not know the structure of the economy. The issue of structural uncertainty in a dynamic game with an investment externality is studied in a companion paper (Mirman and Santugini, 2014). Unlike uncertainty in outcomes, structural uncertainty evolves through learning. In that case, agents make utilization and production decisions as well as learn simultaneously about the stochastic process. Although the characterization of a dynamic game with Bayesian dynamics (and without the assumption of adaptive learning) is generally intractable, we characterize the symmetric Bayesian-learning recursive Cournot-Nash equilibrium. The addition of learning to a stochastic environment is shown to have a profound effect on the equilibrium since decision-making and learning are nonseparable and influence each other.

A Solution of the Social Planner

In this appendix, we derive the social planner's solution in the case of complementary and substitutionary investment choices. We consider the infinite horizon by conjecturing that the value function is of the form $W^{\infty}(y) = \kappa_{\infty} \ln y + \Psi_{\infty}$. As noted, the linear conjecture can be inferred by solving the problem recursively.

Given $W^{\infty}(y) = \kappa_{\infty} \ln y + \Psi_{\infty}$, (62) is rewritten as

$$\begin{aligned} W^{\infty}(y) = \max_{\{e_j, i_j\}_{j=1}^{2}} & \{\ln(e_1 - i_1) + \ln(e_2 - i_2) + \delta\kappa_{\infty}\overline{\eta}_1 \ln i_1 + \delta\kappa_{\infty}\overline{\eta}_2 \ln i_2 \\ & + \delta\kappa_{\infty}\overline{\alpha} \ln(y - e_1 - e_2) + \delta\Psi_{\infty}\} \end{aligned} \tag{76}$$

if investment choices are complementary and

$$\begin{aligned} W^{\infty}(y) = \max_{\{e_j, i_j\}_{j=1}^{2}} & \{\ln(e_1 - i_1) + \ln(e_2 - i_2) + \delta\kappa_{\infty}\overline{\eta} \ln(i_1 + i_2) \\ & + \delta\kappa_{\infty}\overline{\alpha} \ln(y - e_1 - e_2) + \delta\Psi_{\infty}\} \end{aligned} \tag{77}$$

if investment choices are substitutionary.

For complements, for $j = 1, 2$, the first-order conditions corresponding to (76) are

$$e_j : \frac{1}{e_j - i_j} = \frac{\delta\kappa_{\infty}\overline{\alpha}}{y - e_1 - e_2}, \tag{78}$$

$$i_j : \frac{1}{e_j - i_j} = \frac{\delta\kappa_{\infty}\overline{\eta}_j}{i_j}, \tag{79}$$

which yields

$$E_j^{*\infty}(y) = \frac{1 + \delta\kappa_{\infty}\overline{\eta}_j}{2 + \delta\kappa_{\infty}(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha})} y, \tag{80}$$

$$I_j^{*\infty}(y) = \frac{\delta\kappa_{\infty}\overline{\eta}_j}{2 + \delta\kappa_{\infty}(\overline{\eta}_1 + \overline{\eta}_2 + \overline{\alpha})} y. \tag{81}$$

Plugging (80) and (81) back into (76) implies that

$$\kappa_\infty = \frac{2}{1-\delta(\overline{\eta}_1+\overline{\eta}_2+\overline{\alpha})}. \tag{82}$$

Plugging (82) into (80) and (81) and summing over j yields (63) and (65). Plugging (63) and (65) into $\sum_{j=1}^{2} C_j^{*\infty}(y) = \sum_{j=1}^{2}(E_j^{*\infty}(y) - I_j^{*\infty}(y))$ yields (64).

For substitutes, for $j = 1, 2$, the first-order conditions corresponding to (77) are

$$e_j : \frac{1}{e_j - i_j} = \frac{\delta\kappa_\infty\overline{\alpha}}{y - e_1 - e_2} \tag{83}$$

$$i_j : \frac{1}{e_j - i_j} = \frac{\delta\kappa_\infty\overline{\eta}}{i_1 + i_2}, \tag{84}$$

which yields

$$E_j^{*\infty}(y) = \frac{2+\delta\kappa_\infty\overline{\eta}}{4+2\delta\kappa_\infty(\overline{\eta}+\overline{\alpha})}y \tag{85}$$

$$I_1^{*\infty}(y) + I_2^{*\infty}(y) = \frac{\delta\kappa_\infty\overline{\eta}}{2+\delta\kappa_\infty(\overline{\eta}+\overline{\alpha})}y \tag{86}$$

since the social planner only needs to solve for total investment. Plugging (85) and (86) back into (77) yields

$$\kappa_\infty = \frac{2}{1-\delta(\overline{\eta}+\overline{\alpha})}. \tag{87}$$

Plugging (87) into (85) and summing over j yields (66). Plugging (87) into (86) yields (68). Plugging (66) and (68) into $\sum_{j=1}^{2} C_j^{*\infty}(y) = \sum_{j=1}^{2}(E_j^{*\infty}(y) - I_j^{*\infty}(y))$ yields (67).

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