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Learning and Technological Progess in Dynamic Games

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Abstract:
We study investment and consumption decisions in a dynamic game under learning. To that end, we present a model in which agents not only extract a resource for consumption, but also invest in technology to improve the future stock. At the same time, the agents learn about the stochastic process governing the evolution of public capital, including the effect of investment in technology on future stock. Although the characterization of a dynamic game with Bayesian dynamics (and without the assumption of adaptive learning) is generally intractable, we characterize the unique symmetric Bayesian-learning recursive Cournot-Nash equilibrium for any finite horizon and for general distributions of the random variables. We also show that the limits of the equilibrium outcomes for a finite horizon exist. The addition of learning to a stochastic environment is shown to have a profound effect on the equilibrium.

Keywords: Capital accumulation, Dynamic game, Investment, Learning, Risk, Technological progress

JEL Classification: C72, C73, D81, D83, L13, Q20

1 Introduction

The evolution of public capital such as infrastructures, roads, telecommunications, energy, and common-pool natural resources plays a key role in economic analysis. While agents derive utility from the utilization of public capital, they also have the ability to invest in technology to increase the productivity of future stocks of public capital, thereby maintaining or expanding future production and consumption possibilities. Because several agents contribute to the technological progress for the stock of a public capital, there is an externality contained in this investment. Although agents have a certain control over technological progress, the evolution of public capital is highly uncertain. Indeed, the evolution of public capital depends on random shocks, which implies that agents make decisions without knowing the realizations of these shocks, i.e., they face *uncertainty in outcomes*. However, agents generally face more than just uncertainty in outcomes since the true distributions of these shocks are never known exactly. In other words, agents generally face *structural uncertainty* because they do not know the structure of the economy. Unlike uncertainty in outcomes, structural uncertainty evolves through learning. That is, the agents gather and analyze data in order to learn the distribution of the random shocks. In that case, agents make consumption and investment decisions as well as learn simultaneously about the stochastic process. In general, decision-making and learning are nonseparable and influence each other.[1]

It is the purpose of this paper to study endogenous technological progress in a dynamic game of capital utilization with learning. Our model adds two important features by allowing agents to strategically invest in technology, while at the same time learning about the stochastic process governing the evolution of public capital, including the effect of investment in technology on future stock. Strategic utilization of public capital has already been studied without the issue of endogenous technological change and learning, beginning

[1]There is a two-way interaction between decision making and learning. On the one hand, decision making may have an effect on learning, which is referred as experimentation. On the other hand, the presence of learning adds risk which affects future payoffs and thus behavior.

with the Great Fish War model of Levhari and Mirman (1980) in the deterministic case. Recently, Antoniadou et al. (2012) studies strategic exploitation of a common-pool natural resource under uncertainty in outcomes but without the inclusion of technological progress or learning. In their model, the distribution of the random shock is known and the agents have no need to learn about the structure of the economy.[2] The issue of learning has generally been addressed in dynamic single-agent problems, thereby removing the issue of strategic interactions and externalities.[3] In particular, the effect of learning on consumption only (without the inclusion of an investment decision) has already been studied in the context of the Mirman-Zilcha model in Koulovatianos, Mirman, and Santugini (2009).

In our model, agents not only extract a resource for consumption, but also invest to improve the future stock. Hence, the model has both a dynamic externality (i.e., consumption of an agent affects the payoff of the other agent through the evolution of the stock) and an *investment* externality (i.e., investment of one agent has a positive effect on the public good and thus future payoffs of the other agent). Moreover, agents face structural uncertainty because the distributions of the shocks affecting the evolution of the stock are unknown. However, agents observe past shocks and learn using Bayesian methods.[4] Agents are fully rational and anticipate learning, which entwines decision-making with learning.[5] That is, the learning activity due to structural uncertainty is directly embedded in a dynamic game of public capital utilization. The agents' learning activity entwine with the strategic interaction adds an element of risk to the decision making process.

For the Levhari and Mirman (1980) framework with an investment exter-

[2]In a different context, Mirman and To (2005) has addressed the issue of investing in capital in an overlapping generation model.

[3]See Bernhardt and Taub (2011) for a recent paper on learning in oligopoly when the firms learn from prices.

[4]There is no experimentation in our model. For the literature on single-agent experimentation with capital accumulation , see Freixas (1981), Bertocchi and Spagat (1998), Datta et al. (2002), El-Gamal and Sundaram (1993), Huffman and Kiefer (1994), and Beck and Wieland (2002).

[5]There is no adaptive learning. Under adaptive learning, agents are bounded because they assume that beliefs will not change over time, i.e., they do not anticipate learning. See Evans and Honkapohja (2001) for a detailed exposition of adaptive learning.

nality, we derive and characterize the unique symmetric Bayesian-learning recursive Cournot-Nash equilibrium for any finite horizon and for general distribution of the shocks and the beliefs. We also show that the limits of the equilibrium outcomes for a finite horizon exist. Although we assume a log utility and a Cobb-Douglas production function, we do not rely on conjugate families for distribution, i.e., we allow for the prior and posterior p.d.f.'s to belong to different families. This generality is important since the use of conjugate priors are restrictive.[6] In addition to providing a detailed guide to the derivation of the equilibrium under learning when beliefs are anticipated to be updated many times, we study the effect of learning on behavior, both consumption and investment. The addition of learning has a profound change on equilibrium values. Without learning, the Levhari and Mirman (1980) framework (with or without the investment externality) displays certainty equivalence, i.e., the random production shocks affect the optimization problem through their means. When distributions of the shocks are unknown and agents learn about them, there is no certainty equivalence and higher moments of the distribution for beliefs have an effect on the equilibrium values. In particular, we show that the anticipation of learning generates more risk about the future, which induces agents to decrease consumption, increase investment, while overall extraction is ambiguous.

The paper is organized as follows. Section 2 presents the model and defines the Bayesian-learning recursive Cournot-Nash equilibrium. Section 3 derives and characterize the equilibrium for both finite and infinite horizons. Section 4 studies the effect of learning on the equilibrium when beliefs are unbiased.

2 Model and Equilibrium Definition

In this section, we embed learning in a dynamic game in which agents make both consumption and investment decisions and at the same time learn about the stochastic process governing the evolution of the capital. We first present

[6]One popular approach is to rely on the fact that the family of normal distributions with an unknown mean is a conjugate family for samples from a normal distribution.

the model. We then define the Bayesian-learning recursive Cournot-Nash equilibrium.

2.1 Model

Consider the Great Fish War dynamic game of Levhari and Mirman (1980) in which several agents derive utility from the consumption of a common capital stock. Formally, let y_t be capital stock available at the beginning of period t. If the capital goes unexploited in period t, then the stock evolves stochastically according to the rule

$$\tilde{y}_{t+1} = y_t^{\tilde{\eta}_{\beta,t}} \tag{1}$$

where $\tilde{\eta}_{\beta,t}$ is a random shock in period t, i.e., the shock is realized in period $t+1$.[7]

In period t, agent j extracts a quantity $c_{j,t}$ from the stock y_t, which yields utility $u(c_{j,t}) = \ln c_{j,t}$, $j = 1, 2$. In addition to consuming the stock, agent j extracts additional $i_{j,t}$ units that are immediately invested in order to increase the future stock. Hence, a total $\sum_{j=1}^{2} i_{j,t}$ is invested back, and the remaining $y_t - \sum_{j=1}^{2} (c_{j,t} + i_{j,t})$ is left for future use. The present consumption and investment of the capital by the two agents have an effect on the future stock. The evolution of an exploited stock follows the stochastic rule

$$\tilde{y}_{t+1} = \left(\sum_{j=1}^{2} i_j\right)^{\tilde{\eta}_{\alpha,t}} \left(y - \sum_{j=1}^{2} (c_{j,t} + i_{j,t})\right)^{\tilde{\eta}_{\beta,t}} \tag{2}$$

where $\tilde{\boldsymbol{\eta}}_t = [\tilde{\eta}_{\alpha,t}, \tilde{\eta}_{\beta,t}]$ is random iid over time. Let the p.d.f. of $\tilde{\boldsymbol{\eta}}_t$ be $\phi(\boldsymbol{\eta}_t|\theta^*), \boldsymbol{\eta}_t \in \mathbb{R}_+^2$ which depends on $\theta^* \in \Theta \subset \mathbb{R}^N$ for $N \in \mathbb{N}$.

To simplify notation, the t-subscript for indexing time is hereafter removed and the hat sign is used to indicate the value of a variable in the subsequent period, i.e., y is stock today and

$$\hat{y} = \left(\sum_{j=1}^{2} i_j\right)^{\eta_\alpha} \left(y - \sum_{j=1}^{2} (c_j + i_j)\right)^{\eta_\beta} \tag{3}$$

[7] A tilde sign is used to distinguish a random variable from its realization.

is stock tomorrow. From (3), the shocks η_α and η_β measure the contributions of the unexploited stock and the investment goods, respectively, toward the new capital. Specifically, from (3), for a given shock η_α, investment improves the renewability of the stock through the investment component $(i_1 + i_2)^{\eta_\alpha}$ such that investment goods are perfect substitutes.

Having described the stochastic evolution of the stock, we next explain the information available to the agents and their learning process. The agents do not know the value of θ^* (i.e., they do not know the distribution of $\tilde{\boldsymbol{\eta}}$), but they have common prior beliefs about its value expressed as a prior p.d.f. ξ on Θ. That is, the probability that $\theta^* \in S$ is $\int_{\theta \in S} \xi(\theta) \mathrm{d}\theta$ for any $S \subset \Theta$. Because the shock $\boldsymbol{\eta}$ is observable, the agents update their beliefs using Bayesian method. Here, the agents learn about the stochastic process that governs the evolution of the stock. In particular, the agents learn about the impact of their investment on the future stock through the distribution of $\tilde{\eta}_\alpha$. Given a prior ξ and the observation $\boldsymbol{\eta}$ today, the posterior tomorrow is

$$\hat{\xi}(\theta|\boldsymbol{\eta}) = \frac{\phi(\boldsymbol{\eta}|\theta)\xi(\theta)}{\int_{x\in\Theta} \phi(\boldsymbol{\eta}|x)\xi(x)\mathrm{d}x} \tag{4}$$

for $\theta \in \Theta$, by Bayes' Theorem.

To distinguish among different horizons of the dynamic game, we use the index $\tau = 0, 1, \ldots, T$. Given the present stock of the capital, agent j maximizes the expected sum of discounted utilities. The anticipation of acquiring and using data is embedded directly in the value function. Hence, for $j, k = 1, 2, j \neq k$, the τ-period-horizon value function for agent j is

$$\begin{aligned} v_j^\tau(y;\xi) = \max_{c_j, i_j} \Bigg\{ & \ln c_j + \delta \int_{\boldsymbol{\eta}\in\mathbb{R}_+^2} v_j^{\tau-1} \left((i_1+i_2)^{\eta_\alpha} \left(y - \sum\nolimits_{j=1}^2 (c_j + i_j) \right)^{\eta_\beta} ; \hat{\xi}(\cdot|\boldsymbol{\eta}) \right) \\ & \cdot \left[\int_{\theta\in\Theta} \phi(\boldsymbol{\eta}|\theta)\xi(\theta)\mathrm{d}\theta \right] \mathrm{d}\boldsymbol{\eta} \Bigg\}, \end{aligned} \tag{5}$$

where $\int_{\theta\in\Theta} \phi(\boldsymbol{\eta}|\theta)\xi(\theta)\mathrm{d}\theta$ is the expected p.d.f. of the shocks. From (5), learning is anticipated using Bayesian updating. In a dynamic context, rational expectations imply that the information contained in the future production shock is anticipated. The anticipation of learning is integrated into (5) by anticipating the updated beliefs from ξ to $\hat{\xi}(\cdot|\boldsymbol{\eta})$ using (4). As shown by Koulovatianos, Mirman, and Santugini (2009) in a growth single-agent model without investment, the anticipation of learning is related to the nonseparability of control and learning since the dynamics given in (2) and (4) are entwined through the shocks. The anticipation of learning generates more uncertainty for the agents because future beliefs are treated as random variables from today's perspective.

2.2 Equilibrium Definition

We next define the Bayesian-learning recursive Cournot-Nash equilibrium for a T-period-horizon game in the Levhari and Mirman (1980) framework with an investment externality. The equilibrium consists of the strategies of the two agents for every horizon from the first period (when there are T periods left) to the last period (when there is no horizon). Without loss of generality, we assume that the two agents split the stock equally and do not invest in the last period. As in Levhari and Mirman (1980), the assumption of a log utility function implies that the allocation of the stock in the last period has no effect on the dynamic game. Condition 1 states the behavior in the last period, i.e., when the horizon is $\tau = 0$. Condition 2 states the equilibrium for every horizon of the game. Statement 2a is consistent with statement 1. Statement 2b reflects the recursive nature of the equilibrium in which the equilibrium continuation value function for a τ-period horizon depends on the equilibrium strategies for τ'-period horizons, $\tau > \tau' \geq 0$. Learning is embedded in the dynamic game through the anticipation of updated beliefs.

Definition 2.1. *The tuple* $\{C_1^\tau(y;\xi), I_1^\tau(y;\xi), C_2^\tau(y;\xi), I_2^\tau(y;\xi)\}_{\tau=0}^T$ *is a Bayesian-learning recursive Cournot-Nash equilibrium for a* T*-period-horizon game if, for all* y *and* ξ*,*

1. *For* $\tau = 0$, $C_1^0(y;\xi) = C_2^0(y;\xi) = y/2, I_1^0(y;\xi) = I_2^0(y;\xi) = 0$.

2. *For* $\tau = 1, 2, \ldots, T$*, for* $j, k = 1, 2, j \neq k$*, given* $\{C_k^\tau(y;\xi), I_k^\tau(y;\xi)\}$ *and* $\{C_1^t(y;\xi), I_1^t(y;\xi), C_2^t(y;\xi), I_2^t(y;\xi)\}_{t=0}^{\tau-1}$*,*

$$\begin{aligned}
&\{C_j^\tau(y;\xi), I_j^\tau(y;\xi)\} \\
&= \arg\max_{c_j, i_j} \Bigg\{ \ln c_j \\
&+ \delta \int_{\boldsymbol{\eta}\in\mathbb{R}_+^2} V_j^{\tau-1}\left((i_j + I_k^\tau(y;\xi))^{\eta_\alpha} \left(y - c_j - i_j - C_k^\tau(y;\xi) - I_k^\tau(y;\xi)\right)^{\eta_\beta}; \hat{\xi}(\cdot|\boldsymbol{\eta}) \right) \\
&\cdot \left[\int_{\theta\in\Theta} \phi(\boldsymbol{\eta}|\theta)\xi(\theta)\mathrm{d}\theta \right] \mathrm{d}\boldsymbol{\eta} \Bigg\} \qquad (6)
\end{aligned}$$

where the posterior $\hat{\xi}(\cdot|\boldsymbol{\eta})$ *is consistent with* (4)*. Moreover, for any* y' *and* ξ'*,*

(a) *For* $\tau' = 0$*,*

$$V_1^0(y';\xi') = V_2^0(y',\xi') = \ln(y'/2). \qquad (7)$$

(b) *For* $\tau' = 1, 2, \ldots, T-1$*,*

$$V_j^{\tau'}(y';\xi') = \ln C_j^{\tau'}(y';\xi') + \delta \int_{\boldsymbol{\eta}'\in\mathbb{R}_+^2} V_j^{\tau'-1}\left(\Gamma; \hat{\xi}'(\cdot|\boldsymbol{\eta}')\right) \left[\int_{\theta'\in\Theta} \phi(\boldsymbol{\eta}'|\theta')\xi'(\theta')\mathrm{d}\theta' \right] \mathrm{d}\boldsymbol{\eta}', \qquad (8)$$

such that $\hat{\xi}'(\cdot|\boldsymbol{\eta}')$ *is consistent with* (4) *and*

$$\begin{aligned}
\Gamma \equiv &\left(\sum_{s=1}^2 I_s^{\tau'-1}(y';\xi') \right)^{\eta'_\alpha} \\
&\cdot \left(y' - \sum_{s=1}^2 \left(C_s^{\tau'-1}(y';\xi') + I_s^{\tau'-1}(y';\xi') \right) \right)^{\eta'_\beta}. \qquad (9)
\end{aligned}$$

3 Equilibrium Characterization

In this section, using Definition 2.1, we fully characterize the symmetric equilibrium for any finite horizon.[8] We also show that the limit of the finite-horizon equilibrium exists. We provide a detailed guide to the derivation of the equilibrium and explain how to deal with the fact that the continuation value function encompasses beliefs that have been updated many times. We finally give several examples to show the generality of our results.

Proposition 3.1 provides the equilibrium for any finite horizon. The equilibrium values hold for general distributions, i.e., no assumption is needed on the production shock as well as on the distribution of prior beliefs beyond the fact that integrals must exist. The model does away with all the difficulties inherent in Bayesian analysis. In particular, the prior need not belong to the conjugate family of the distribution of the production shock. In other words, expression for equilibrium consumption and investment are valid for a wide range of priors, even those that are outside of families of distributions that are closed under sampling. Note that the equilibrium values depend on the mean of the shocks (conditional on the unknown parameter) and the distribution of beliefs. In other words, the conditional mean shocks

$$\mu_s(\tilde{\theta}) \equiv \int_{\boldsymbol{\eta} \in \mathbb{R}_+^2} \eta_s \phi(\boldsymbol{\eta}|\tilde{\theta}) \mathrm{d}\boldsymbol{\eta}, \tag{10}$$

$s \in \{\alpha, \beta\}$ are random variables because $\tilde{\theta}$ is unknown and random from the agents' point of view.

Proposition 3.1. *Suppose that for $0 < \mu_\alpha(\theta) + \mu_\beta(\theta) < \infty$ for all $\theta \in \Theta$. Then, there exists a unique symmetric Bayesian-learning recursive Cournot-Nash equilibrium for a T-period game, $T = 1, 2, \ldots$ In equilibrium, for $\tau =$*

[8]Since the investment goods are perfectly substitutable, there exists many equilibrium points in which only total investment can be determined. However, for any equilibrium and for any horizon, total investment remains the same whereas individual investment changes.

$0, 1, \ldots, T$, *each agent consumes*

$$C^{\tau}(y;\xi) = \left(\int\limits_{\theta\in\Theta} \frac{2-\delta(\mu_\alpha(\theta)+\mu_\beta(\theta))(1+\delta^{\tau}(\mu_\alpha(\theta)+\mu_\beta(\theta))^{\tau})}{1-\delta(\mu_\alpha(\theta)+\mu_\beta(\theta)} \xi(\theta)\mathrm{d}\theta \right)^{-1} y, \tag{11}$$

and invests

$$I^{\tau}(y;\xi) = \frac{\delta \int\limits_{\theta\in\Theta} \frac{\mu_\alpha(\theta)(1-\delta^{\tau}(\mu_\alpha(\theta)+\mu_\beta(\theta))^{\tau})}{1-\delta(\mu_\alpha(\theta)+\mu_\beta(\theta))}\xi(\theta)\mathrm{d}\theta}{2\int\limits_{\theta\in\Theta} \frac{2-\delta(\mu_\alpha(\theta)+\mu_\beta(\theta))(1+\delta^{\tau}(\mu_\alpha(\theta)+\mu_\beta(\theta))^{\tau}))}{1-\delta(\mu_\alpha(\theta)+\mu_\beta(\theta))}\xi(\theta)\mathrm{d}\theta} y, \tag{12}$$

for $s \in \{\alpha, \beta\}$,

Proof. We first derive equilibrium consumption, investment, and value function in the one-period horizon. We then consider a τ-period horizon and solve for equilibrium consumption, investment and value functions recursively. We finally impose the initial condition given by the equilibrium for the one-period horizon to characterize the equilibrium for any horizon. To simplify notation in the proof, let $\phi^e(\boldsymbol{\eta}) \equiv \int\limits_{\theta\in\Theta} \phi(\boldsymbol{\eta}|\theta)\xi(\theta)\mathrm{d}\theta$.

1. Consider first the one-period-horizon game. Using (6) and (7), for $j,k = 1,2, j \neq k$, given $\{C^1(y;\xi), I^1(y;\xi)\}$, agent j's one-period-horizon optimal policies satisfy

$$\begin{aligned}&\{C^1(y;\xi), I^1(y;\xi)\}\\ &= \arg\max_{c_j,i_j} \left\{ \ln c_j + \delta \left(\int\limits_{\theta\in\Theta} \mu_\alpha(\theta)\xi(\theta)\mathrm{d}\theta \right) \ln(i_j + I^1(y;\xi)) \right.\\ &\qquad \left. + \delta \left(\int\limits_{\theta\in\Theta} \mu_\beta(\theta)\xi(\theta)\mathrm{d}\theta \right) \ln(y - c_j - i_j - C^1(y;\xi) - I^1(y;\xi)) + \delta \ln 2 \right\},\end{aligned} \tag{13}$$

$\mu_s(\theta) \equiv \int\limits_{\boldsymbol{\eta}\in\mathbb{R}_+} \eta_s \phi(\boldsymbol{\eta}|\theta)\mathrm{d}\boldsymbol{\eta}$, $s \in \{\alpha,\beta\}$. The first-order conditions corresponding to (13) are

$$c_j : \frac{1}{c_j} = \frac{\delta \int\limits_{\theta\in\Theta} \mu_\beta(\theta)\xi(\theta)\mathrm{d}\theta}{y - c_j - i_j - C^1(y;\xi) - I^1(y;\xi)}, \tag{14}$$

$$i_j : \frac{\delta \int\limits_{\theta\in\Theta} \mu_\alpha(\theta)\xi(\theta)\mathrm{d}\theta}{i_j + I^1(y;\xi)} = \frac{\delta \int\limits_{\theta\in\Theta} \mu_\beta(\theta)\xi(\theta)\mathrm{d}\theta}{y - c_j - i_j - C^1(y;\xi) - I^1(y;\xi)}, \tag{15}$$

evaluated at $c_j = C^1(y;\xi)$ and $i_j = I^1(y;\xi)$. Rearranging (14) and (15) yields

$$C^1(y;\xi) = \frac{y - 2I^1(y;\xi)}{2 + \delta \int\limits_{\theta\in\Theta} \mu_\beta(\theta)\xi(\theta)\mathrm{d}\theta}, \tag{16}$$

and

$$I^1(y;\xi) = \frac{(y - 2C^1(y;\xi)) \int\limits_{\theta\in\Theta} \mu_\alpha(\theta)\xi(\theta)\mathrm{d}\theta}{2 \int\limits_{\theta\in\Theta} (\mu_\alpha(\theta) + \mu_\beta(\theta))\xi(\theta)\mathrm{d}\theta}. \tag{17}$$

Solving (16) and (17) for equilibrium one-period-horizon consumption and investment yields a unique solution:

$$C^1(y;\xi) = \frac{y}{2 + \delta \int\limits_{\theta\in\Theta} (\mu_\alpha(\theta) + \mu_\beta(\theta))\xi(\theta)\mathrm{d}\theta}, \tag{18}$$

$$I^1(y;\xi) = \frac{\delta(\int\limits_{\theta\in\Theta} \mu_\alpha(\theta)\xi(\theta)\mathrm{d}\theta)y}{2(2 + \delta \int\limits_{\theta\in\Theta} (\mu_\alpha(\theta) + \mu_\beta(\theta))\xi(\theta)\mathrm{d}\theta)}. \tag{19}$$

Plugging (18) and (19) into the objective function in (13) yields

$$V^1(y;\xi) = \left(1 + \int\limits_{\theta\in\Theta} (\mu_\alpha(\theta) + \mu_\beta(\theta))\xi(\theta)\mathrm{d}\theta\right) \ln y + \Psi^1(\xi), \tag{20}$$

where $\Psi^1(\xi)$ depends on beliefs but not on the stock.

2. Having solved for the one-period-horizon, we consider next a τ-period-horizon where the continuation value function is $V^{\tau-1}(y;\xi) = \kappa^{\tau-1}(\xi)\ln y + \Psi^{\tau-1}(\xi)$ where $\kappa^{\tau-1}(\xi)$ and $\Psi^{\tau-1}(\xi)$ are unknown functions of ξ. Given

$V^{\tau-1}(y;\xi) = \kappa^{\tau-1}(\xi)\ln y + \Psi^{\tau-1}(\xi)$ and $\{C^{\tau}(y;\xi), I^{\tau}(y;\xi)\}$, agent j's τ-period-horizon optimal policies satisfy

$$
\begin{aligned}
&\{C^1(y;\xi), I^1(y;\xi)\} \\
&= \arg\max_{c_j, i_j} \Biggl\{ \ln c_j + \delta \left(\int\limits_{\boldsymbol{\eta}\in\mathbb{R}_+^2} \kappa^{\tau-1}(\hat{\xi}(\cdot|\boldsymbol{\eta}))\eta_\alpha \phi^e(\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta} \right) \ln(i_j + I^{\tau}(y;\xi)) \\
&\quad + \delta \left(\int\limits_{\boldsymbol{\eta}\in\mathbb{R}_+^2} \kappa^{\tau-1}(\hat{\xi}(\cdot|\boldsymbol{\eta}))\eta_\beta \phi^e(\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta} \right) \ln(y - c_j - i_j - C^{\tau}(y;\xi) - I^{\tau}(y;\xi)) \\
&\quad + \delta \int\limits_{\boldsymbol{\eta}\in\mathbb{R}_+^2} \Psi^{\tau-1}(\hat{\xi}(\cdot|\boldsymbol{\eta}))\phi^e(\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta} \Biggr\}. \qquad (21)
\end{aligned}
$$

The first-order conditions corresponding to (21) are

$$
c_j : \frac{1}{c_j} = \frac{\delta \int\limits_{\boldsymbol{\eta}\in\mathbb{R}_+^2} \kappa^{\tau-1}(\hat{\xi}(\cdot|\boldsymbol{\eta}))\eta_\beta \phi^e(\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta}}{y - c_j - i_j - C^{\tau}(y;\xi) - I^{\tau}(y;\xi)}, \qquad (22)
$$

$$
i_j : \frac{\delta \int\limits_{\boldsymbol{\eta}\in\mathbb{R}_+^2} \kappa^{\tau-1}(\hat{\xi}(\cdot|\boldsymbol{\eta}))\eta_\alpha \phi^e(\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta}}{i_j + I^{\tau}(y;\xi)} = \frac{\delta \int\limits_{\boldsymbol{\eta}\in\mathbb{R}_+^2} \kappa^{\tau-1}(\hat{\xi}(\cdot|\boldsymbol{\eta}))\eta_\beta \phi^e(\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta}}{y - c_j - i_j - C^{\tau}(y;\xi) - I^{\tau}(y;\xi)} \qquad (23)
$$

evaluated at $c_j = C^{\tau}(y;\xi)$ and $i_j = I^{\tau}(y;\xi)$. From (22) and (23), $C^{\tau}(y;\xi)$ and $I^{\tau}(y;\xi)$ are linear in y. Solving (22) and (23) yields a unique solution:

$$C^{\tau}(y;\xi) = \frac{y}{2+\delta \int\limits_{\boldsymbol{\eta}\in\mathbb{R}_+^2} \kappa^{\tau-1}(\hat{\xi}(\cdot|\boldsymbol{\eta}))(\eta_\alpha+\eta_\beta)\phi^e(\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta}}, \tag{24}$$

$$I^{\tau}(y;\xi) = \frac{\delta\left(\int\limits_{\boldsymbol{\eta}\in\mathbb{R}_+^2} \kappa^{\tau-1}(\hat{\xi}(\cdot|\boldsymbol{\eta}))\eta_\alpha\phi^e(\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta}\right)y}{2\left(2+\delta \int\limits_{\boldsymbol{\eta}\in\mathbb{R}_+^2} \kappa^{\tau-1}(\hat{\xi}(\cdot|\boldsymbol{\eta}))(\eta_\alpha+\eta_\beta)\phi^e(\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta}\right)}. \tag{25}$$

Plugging (24) and (25) into the objective function in (21) yields

$$\begin{aligned} V^{\tau}(y;\xi) &= \left(1+\delta \int\limits_{\boldsymbol{\eta}\in\mathbb{R}_+^2} \kappa^{\tau-1}(\hat{\xi}(\cdot|\boldsymbol{\eta}))(\eta_\alpha+\eta_\beta)\phi^e(\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta}\right)\ln y \\ &\quad + \delta \int\limits_{\boldsymbol{\eta}\in\mathbb{R}_+^2} \Psi_j^{\tau-1}(\hat{\xi}(\cdot|\boldsymbol{\eta}))\phi^e(\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta} + \Gamma(\xi), &(26) \\ &\equiv \kappa^{\tau}(\xi)\ln y + \Psi^{\tau}(\xi), &(27) \end{aligned}$$

where $\Gamma(\xi)$ is a cumbersome function of ξ that has no effect on the equilibrium and is not characterized here. Hence,

$$\kappa^{\tau}(\xi) = 1+\delta \int\limits_{\boldsymbol{\eta}\in\mathbb{R}_+^2} \kappa^{\tau-1}(\hat{\xi}(\cdot|\boldsymbol{\eta}))(\eta_\alpha+\eta_\beta)\phi^e(\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta} \tag{28}$$

with, from (20), initial condition

$$\begin{aligned} \kappa^{1}(\xi) &= 1+\delta \int\limits_{\theta\in\Theta} (\mu_\alpha(\theta)+\mu_\beta(\theta))\xi(\theta)\mathrm{d}\theta, &(29) \\ &= 1+\delta \int\limits_{\boldsymbol{\eta}\in\mathbb{R}_+^2} (\eta_\alpha+\eta_\beta)\phi^e(\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta}. &(30) \end{aligned}$$

where recall that $\mu_s(\theta) \equiv \int\limits_{\boldsymbol{\eta}\in\mathbb{R}_+} \eta_s\phi(\boldsymbol{\eta}|\theta)\mathrm{d}\boldsymbol{\eta}, s \in \{\alpha,\beta\}$. From (28) and (30), it follows that

$$\kappa^{\tau}(\xi) = \int\limits_{\theta\in\Theta} \sum_{t=0}^{\tau} \delta^t(\mu_\alpha(\theta) + \mu_\beta(\theta))^t \xi(\theta)\mathrm{d}\theta. \tag{31}$$

Plugging (4) and (31) into (24) yields

$$\begin{aligned}
&C^{\tau}(y;\xi) \\
&= \left(2 + \delta \int\limits_{\boldsymbol{\eta}\in\mathbb{R}_+^2} \left(\int\limits_{\theta\in\Theta} \sum_{t=0}^{\tau-1} \delta^t(\mu_\alpha(\theta) + \mu_\beta(\theta))^t \frac{\phi(\boldsymbol{\eta}|\theta)\xi(\theta)}{\int_{x\in\Theta}\phi(\boldsymbol{\eta}|x)\xi(x)\mathrm{d}x}\mathrm{d}\theta\right)\right. \\
&\quad \left. \cdot\, (\eta_\alpha + \eta_\beta)\left(\int_{x\in\Theta}\phi(\boldsymbol{\eta}|x)\xi(x)\mathrm{d}x\right)\mathrm{d}\boldsymbol{\eta}\right)^{-1} y, \qquad (32) \\
&= \frac{y}{2 + \delta \int\limits_{\theta\in\Theta} \sum\limits_{t=0}^{\tau-1} \delta^t(\mu_\alpha(\theta) + \mu_\beta(\theta))^t \int\limits_{\boldsymbol{\eta}\in\mathbb{R}_+^2} (\eta_\alpha + \eta_\beta)\phi(\boldsymbol{\eta}|\theta)\mathrm{d}\boldsymbol{\eta}\xi(\theta)\mathrm{d}\theta}, \qquad (33) \\
&= \frac{y}{2 + \delta \int\limits_{\theta\in\Theta} \sum\limits_{t=0}^{\tau-1} \delta^t(\mu_\alpha(\theta) + \mu_\beta(\theta))^t(\mu_\alpha(\theta) + \mu_\beta(\theta))\xi(\theta)\mathrm{d}\theta}, \qquad (34) \\
&= \frac{y}{\int\limits_{\theta\in\Theta} \frac{2-\delta(\mu_\alpha(\theta)+\mu_\beta(\theta))(1+\delta^{\tau}(\mu_\alpha(\theta)+\mu_\beta(\theta))^{\tau})}{1-\delta(\mu_\alpha(\theta)+\mu_\beta(\theta))}\xi(\theta)\mathrm{d}\theta}, \qquad (35)
\end{aligned}$$

as in (11). Similarly, plugging (4) and (31) into (25) yields

$$I^{\tau}(y;\xi) \tag{36}$$

$$= \frac{\delta\left(\int\limits_{\boldsymbol{\eta}\in\mathbb{R}_+^2}\left(\int\limits_{\theta\in\Theta}\frac{\sum\limits_{t=0}^{\tau-1}\delta^t(\mu_\alpha(\theta)+\mu_\beta(\theta))^t\phi(\boldsymbol{\eta}|\theta)\xi(\theta)}{\int_{x\in\Theta}\phi(\boldsymbol{\eta}|x)\xi(x)\mathrm{d}x}\mathrm{d}\theta\right)\eta_\alpha\int_{x\in\Theta}\phi(\boldsymbol{\eta}|x)\xi(x)\mathrm{d}x\mathrm{d}\boldsymbol{\eta}\right)y}{2\left(2+\delta\int\limits_{\boldsymbol{\eta}\in\mathbb{R}_+^2}\left(\int\limits_{\theta\in\Theta}\frac{\sum\limits_{t=0}^{\tau-1}\delta^t(\mu_\alpha(\theta)+\mu_\beta(\theta))^t\phi(\boldsymbol{\eta}|\theta)\xi(\theta)}{\int_{x\in\Theta}\phi(\boldsymbol{\eta}|x)\xi(x)\mathrm{d}x}\mathrm{d}\theta\right)(\eta_\alpha+\eta_\beta)\int_{x\in\Theta}\phi(\boldsymbol{\eta}|x)\xi(x)\mathrm{d}x\mathrm{d}\boldsymbol{\eta}\right)},$$

$$= \frac{\delta\left(\int\limits_{\theta\in\Theta}\sum\limits_{t=0}^{\tau-1}\delta^t(\mu_\alpha(\theta)+\mu_\beta(\theta))^t\int\limits_{\boldsymbol{\eta}\in\mathbb{R}_+^2}\phi(\boldsymbol{\eta}|\theta)\eta_\alpha\mathrm{d}\boldsymbol{\eta}\xi(\theta)\mathrm{d}\theta\right)y}{2\left(2+\delta\int\limits_{\theta\in\Theta}\sum\limits_{t=0}^{\tau-1}\delta^t(\mu_\alpha(\theta)+\mu_\beta(\theta))^t\int\limits_{\boldsymbol{\eta}\in\mathbb{R}_+^2}(\eta_\alpha+\eta_\beta)\phi(\boldsymbol{\eta}|\theta)\mathrm{d}\boldsymbol{\eta}\xi(\theta)\mathrm{d}\theta\right)}, \tag{37}$$

$$= \frac{\delta\left(\int\limits_{\theta\in\Theta}\sum\limits_{t=0}^{\tau-1}\delta^t(\mu_\alpha(\theta)+\mu_\beta(\theta))^t\mu_\alpha(\theta)\xi(\theta)\mathrm{d}\theta\right)y}{2\left(2+\delta\int\limits_{\theta\in\Theta}\sum\limits_{t=0}^{\tau-1}\delta^t(\mu_\alpha(\theta)+\mu_\beta(\theta))^t(\mu_\alpha(\theta)+\mu_\beta(\theta))\xi(\theta)\mathrm{d}\theta\right)}, \tag{38, 39}$$

$$= \frac{\delta\int\limits_{\theta\in\Theta}\mu_\alpha(\theta)\frac{1-\delta^\tau(\mu_\alpha(\theta)+\mu_\beta(\theta))^\tau}{1-\delta(\mu_\alpha(\theta)+\mu_\beta(\theta))}\xi(\theta)\mathrm{d}\theta}{2\int\limits_{\theta\in\Theta}\frac{2-\delta(\mu_\alpha(\theta)+\mu_\beta(\theta))\left(1+\delta^\tau(\mu_\alpha(\theta)+\mu_\beta(\theta))^\tau\right)}{1-\delta(\mu_\alpha(\theta)+\mu_\beta(\theta))}\xi(\theta)\mathrm{d}\theta}y, \tag{40}$$

as in (12).

□

Proposition 3.2 provides the limits of the equilibrium outcomes stated in Proposition 3.1. As in the finite-horizon games, the random production shock enters the optimization problem through its mean. In other words, the model displays *conditional* certainty equivalence, i.e., only the mean of the shocks conditional on the parameter θ affects consumption and investment. However, the whole distribution defining prior beliefs have an effect on behavior.

Proposition 3.2. *Suppose that* $0 < \mu_\alpha(\theta) + \mu_\beta(\theta) < 1$ *for all* $\theta \in \Theta$*. Then, from* (11) *and* (12), $\lim_{T\to\infty} C^T(y;\xi) \equiv C^\infty(y;\xi)$ *and* $\lim_{T\to\infty} I^T(y;\xi) \equiv I^\infty(y;\xi)$ *exists. In the infinite horizon, each agent consumes*

$$C^\infty(y;\xi) = \left(\int\limits_{\theta\in\Theta} \frac{2-\delta(\mu_\alpha(\theta)+\mu_\beta(\theta))}{1-\delta(\mu_\alpha(\theta)+\mu_\beta(\theta)} \xi(\theta)\mathrm{d}\theta \right)^{-1} y, \tag{41}$$

and invests

$$I^\infty(y;\xi) = \frac{\delta \int\limits_{\theta\in\Theta} \frac{\mu_\alpha(\theta)}{1-\delta(\mu_\alpha(\theta)+\mu_\beta(\theta))}\xi(\theta)\mathrm{d}\theta}{2\int\limits_{\theta\in\Theta} \frac{2-\delta(\mu_\alpha(\theta)+\mu_\beta(\theta))}{1-\delta(\mu_\alpha(\theta)+\mu_\beta(\theta))}\xi(\theta)\mathrm{d}\theta} y. \tag{42}$$

Proof. Taking limits of (11) and (12) yields (41) and (42). □

We present three examples that show the wide applicability of our model, not only in terms of distributions, but also in terms of general unknown structures. For instance, normal distributions are not needed to get analytic results. In Example 3.3, the case of learning about the contribution of investment but learns about several parameters of the distribution. Example 3.4 deals with a uniform distribution for $\tilde{\eta}$ with unknown support. Example 3.5 illustrates the case in which the learning planner does not know to which family $\tilde{\eta}$ belongs, as well as not knowing the parameters characterizing each family.

Example 3.3. *Let* $\tilde{\eta}_\alpha$ *have a beta distribution with unknown parameters* $\theta = (\theta_1, \theta_2)$*, and beliefs* $\xi(\theta_1,\theta_2), \theta_1, \theta_2 > 0$*. Then,* $\mu_\alpha(\theta) = \theta_1/(\theta_1+\theta_2)$*. Let* $\mu_\beta(\theta) = \mu_\beta$ *be independent of* θ *and thus known. Hence,*

$$C^\infty(y;\xi) = \left(\int_{\mathbb{R}^2_{++}} \frac{2-\delta(\theta_1/(\theta_1+\theta_2)+\mu_\beta)}{1-\delta(\theta_1/(\theta_1+\theta_2)+\mu_\beta)} \xi(\theta_1,\theta_2)\mathrm{d}\theta_1\mathrm{d}\theta_2 \right)^{-1} y, \tag{43}$$

$$I^\infty(y;\xi) = \frac{\delta \int_{\mathbb{R}^2_{++}} \frac{\theta_1/(\theta_1+\theta_2)}{1-\delta(\theta_1/(\theta_1+\theta_2)+\mu_\beta)}\xi(\theta_1,\theta_2)\mathrm{d}\theta_1\mathrm{d}\theta_2}{2\int_{\mathbb{R}^2_{++}} \frac{2-\delta(\theta_1/(\theta_1+\theta_2)+\mu_\beta)}{1-\delta(\theta_1/(\theta_1+\theta_2)+\mu_\beta)}\xi(\theta_1,\theta_2)\mathrm{d}\theta_1\mathrm{d}\theta_2} y. \tag{44}$$

Example 3.4. *Let $\tilde{\eta}$ have a uniform distribution with unknown support $[0, \theta_\alpha]$ and $[0, \theta_\beta]$, and beliefs $\xi(\theta_\alpha, \theta_\beta), \theta_\alpha, \theta_\beta \in (0, 1)$. Then, $\mu_\alpha(\theta_\alpha) = \theta_\alpha/2$ and $\mu_\beta(\theta_\beta) = \theta_\beta/2$. Hence,*

$$C^\infty(y;\xi) = \left(\int_0^1 \int_0^1 \frac{4 - \delta(\theta_\alpha + \theta_\beta)}{2 - \delta(\theta_\alpha + \theta_\beta)} \xi(\theta_\alpha, \theta_\beta) \mathrm{d}\theta_\alpha \mathrm{d}\theta_\beta \right)^{-1} y, \tag{45}$$

$$I^\infty(y;\xi) = \frac{\delta \int_0^1 \int_0^1 \frac{\theta_\alpha/2}{1-\delta(\theta_\alpha/2+\theta_\beta)} \xi(\theta_\alpha, \theta_\beta) \mathrm{d}\theta_\alpha \mathrm{d}\theta_\beta}{2 \int_0^1 \int_0^1 \frac{4-\delta(\theta_\alpha+\theta_\beta)}{2-\delta(\theta_\alpha+\theta_\beta)} \xi(\theta_\alpha, \theta_\beta) \mathrm{d}\theta_\alpha \mathrm{d}\theta_\beta} y. \tag{46}$$

Example 3.5. *Let $\Theta = \{\theta_1, \theta_2\}$ where θ_1 and θ_2 refer to two distinct families of distributions. If $0 \leq \rho \leq 1$ is the prior probability that the production shock has distribution θ_1, then*

$$\begin{aligned} C^\infty(y;\xi) = \Bigg(&\rho \int\limits_{\theta_1 \in \Theta} \frac{2 - \delta(\mu_\alpha(\theta_1) + \mu_\beta(\theta_1))}{1 - \delta(\mu_\alpha(\theta_1) + \mu_\beta(\theta_1)} \xi(\theta_1) \mathrm{d}\theta_1 \\ &+ (1-\rho) \int\limits_{\theta_2 \in \Theta} \frac{2 - \delta(\mu_\alpha(\theta_2) + \mu_\beta(\theta_2))}{1 - \delta(\mu_\alpha(\theta_2) + \mu_\beta(\theta_2)} \xi(\theta_2) \mathrm{d}\theta_2 \Bigg)^{-1} y, \end{aligned} \tag{47}$$

$$I^\infty(y;\xi) = \frac{\delta}{2} \left(\rho \frac{\int\limits_{\theta_1 \in \Theta} \frac{\mu_\alpha(\theta_1)}{1-\delta(\mu_\alpha(\theta_1)+\mu_\beta(\theta_1))} \xi(\theta_1) \mathrm{d}\theta_1}{\int\limits_{\theta_1 \in \Theta} \frac{2-\delta(\mu_\alpha(\theta_1)+\mu_\beta(\theta_1))}{1-\delta(\mu_\alpha(\theta_1)+\mu_\beta(\theta_1))} \xi(\theta_1) \mathrm{d}\theta_1} + (1-\rho) \frac{\int\limits_{\theta_1 \in \Theta} \frac{\mu_\alpha(\theta_2)}{1-\delta(\mu_\alpha(\theta_2)+\mu_\beta(\theta_2))} \xi(\theta_2) \mathrm{d}\theta_2}{\int\limits_{\theta_1 \in \Theta} \frac{2-\delta(\mu_\alpha(\theta_2)+\mu_\beta(\theta_2))}{1-\delta(\mu_\alpha(\theta_2)+\mu_\beta(\theta_2))} \xi(\theta_2) \mathrm{d}\theta_2} \right) y. \tag{48}$$

4 Analysis

Having characterized the equilibrium for general distributions of the random variables and provided some examples, we study the effect of learning on the equilibrium. To that end, we compare equilibrium outcomes under learning as defined by (41) and (42) with their counterparts under full information, i.e., θ^* is known by both agents. Note that using the equilibrium outcomes for a finite horizon yields the same comparisons.

Before proceeding with the comparison, we state the equilibrium outcomes under full information. Since our model encompasses the case of informed agents with degenerate beliefs, Proposition 4.1 presents the special case in which the agents knows the distribution of $\tilde{\boldsymbol{\eta}}$. When there is no learning, the equilibrium solution displays certainty equivalence, and there is no substantial effect of uncertainty on behavior, i.e., higher moments of the random shocks do not influence the equilibrium.

Proposition 4.1. *Let the beliefs be degenerate at θ^*. Then,*

$$C^\infty(y;\theta^*) = \frac{1-\delta(\mu_\alpha(\theta^*)+\mu_\beta(\theta^*)}{2-\delta(\mu_\alpha(\theta^*)+\mu_\beta(\theta^*))}y, \tag{49}$$

$$I^\infty(y;\theta^*) = \frac{\delta\mu_\alpha(\theta^*)}{2(2-\delta(\mu_\alpha(\theta^*)+\mu_\beta(\theta^*)))}y. \tag{50}$$

Proof. Evaluating (41) and (42) at the true distribution (i.e., θ^* is known) yields (49) and (50). □

Proposition 4.2 states that learning with unbiased beliefs about the shocks induces both agents to consume less. As in the single-agent case studied in Koulovatianos, Mirman, and Santugini (2009), the risk emanating from learning increases the marginal cost of extraction, which leads to less consumption.

Proposition 4.2. *Suppose that beliefs about the sum of the mean shocks is unbiased, i.e., $\mu_\alpha(\theta^*)+\mu_\beta(\theta^*) = \int_{\theta\in\Theta}(\mu_\alpha(\theta)+\mu_\beta(\theta))\xi(\theta)\mathrm{d}\theta$. Then, learning (with nondegenerate beliefs) decreases consumption, i.e.,*

$$C^\infty(y;\xi) < C^\infty(y;\theta^*). \tag{51}$$

Proof. Applying Jensen's inequality on (41) and (49) yields (51). □

We next turn to the effect of learning on investment. To understand how learning influences behavior, we compare the first-order conditions under learning and full information. Under learning, the value function for any agent evaluated at the equilibrium continuation value function is[9]

$$V(y,\xi) = \max_{c_j,i_j} \ln c_j + \delta \int_{\theta\in\Theta} \frac{\mu_\alpha(\theta)}{1-\delta(\mu_\alpha(\theta)+\mu_\beta(\theta))}\xi(\theta)\mathrm{d}\theta \ln(i_1+i_2)$$
$$+ \delta \int_{\theta\in\Theta} \frac{\mu_\beta(\theta)}{1-\delta(\mu_\alpha(\theta)+\mu_\beta(\theta))}\xi(\theta)\mathrm{d}\theta \ln(y-c_1-i_1-c_2-i_2). \tag{52}$$

In equilibrium, $I^\infty(y;\xi)$ equates the marginal benefit of investing with the marginal cost of extraction, i.e.,

$$\frac{\int_{\theta\in\Theta} \frac{\mu_\alpha(\theta)}{1-\delta(\mu_\alpha(\theta)+\mu_\beta(\theta))}\xi(\theta)\mathrm{d}\theta}{2I^\infty(y;\xi)} = \frac{\int_{\theta\in\Theta} \frac{\mu_\beta(\theta)}{1-\delta(\mu_\alpha(\theta)+\mu_\beta(\theta))}\xi(\theta)\mathrm{d}\theta}{y-2C^\infty(y;\xi)-2I^\infty(y;\xi)}. \tag{53}$$

Under full information, the value function for any agent evaluated at the equilibrium continuation value function is[10]

$$V(y,\xi) = \max_{c_j,i_j} \ln c_j + \delta \frac{\mu_\alpha(\theta^*)}{1-\delta(\mu_\alpha(\theta^*)+\mu_\beta(\theta^*))}\xi(\theta)\mathrm{d}\theta \ln(i_1+i_2)$$
$$+ \delta \frac{\mu_\beta(\theta^*)}{1-\delta(\mu_\alpha(\theta^*)+\mu_\beta(\theta^*))} \ln(y-c_1-i_1-c_2-i_2). \tag{54}$$

In equilibrium, $I^\infty(y;\theta^*)$ equates the marginal benefit of investing with the marginal cost of extraction, i.e.,

$$\frac{\frac{\mu_\alpha(\theta^*)}{1-\delta(\mu_\alpha(\theta^*)+\mu_\beta(\theta^*))}}{2I^\infty(y;\theta^*)} = \frac{\frac{\mu_\beta(\theta^*)}{1-\delta(\mu_\alpha(\theta^*)+\mu_\beta(\theta^*))}}{y-2C^\infty(y;\theta^*)-2I^\infty(y;\theta^*)}. \tag{55}$$

To clarify the discussion, we consider two special cases of structural uncertainty. In the first case, the agents do not know the stochastic process governing the contribution of the investment goods toward new capital, i.e., from (3), the agents know the distribution of $\tilde{\eta}_\beta$ and learn about the distribution of $\tilde{\eta}_\alpha$. In the second case, the agents do not know the stochastic process governing the contribution of the unexploited stock toward new capital.

Suppose first that μ_β is independent of θ and thus known. Suppose fur-

[9]Plugging (31) into (21) (ignoring the constant) and taking limits yields (52).
[10]Evaluating (52) at the true distribution (i.e., θ^* is known) yields (54).

ther that beliefs about the distribution of $\tilde{\eta}_\alpha$ are unbiased, i.e., $\mu_\alpha(\theta^*) = \int_{\theta\in\Theta} \mu_\alpha(\theta)\xi(\theta)\mathrm{d}\theta$. From (53) and (55), the effect of learning is three-fold. First, learning increases the marginal benefit of investing, i.e., by Jensen's inequality,

$$\int_{\theta\in\Theta} \frac{\mu_\alpha(\theta)}{1-\delta(\mu_\alpha(\theta)+\mu_\beta)}\xi(\theta)\mathrm{d}\theta > \frac{\mu_\alpha(\theta^*)}{1-\delta(\mu_\alpha(\theta^*)+\mu_\beta)}. \tag{56}$$

Second, learning increases the marginal cost of investing directly, i.e.,

$$\int_{\theta\in\Theta} \frac{\mu_\beta}{1-\delta(\mu_\alpha(\theta)+\mu_\beta)}\xi(\theta)\mathrm{d}\theta > \frac{\mu_\beta}{1-\delta(\mu_\alpha(\theta^*)+\mu_\beta)}, \tag{57}$$

by Jensen's inequality. Learning also increases the marginal cost of extraction indirectly through lower consumption, i.e., from Proposition 4.2, $C^\infty(y;\xi) < C^\infty(y;\theta^*)$. While there are partial effects that pull in opposite directions, Proposition 4.3 states that the overall effect of learning increases investment.

Proposition 4.3. *Suppose that μ_β is independent of θ and that $\mu_\alpha(\theta^*) = \int_{\theta\in\Theta} \mu_\alpha(\theta)\xi(\theta)\mathrm{d}\theta$. Then, learning increases investment, i.e.,*

$$I^\infty(y;\xi) > I^\infty(y;\theta^*). \tag{58}$$

Proof. The proof applies Jensen's and Holdër's inequalities. Specifically, from (42) and (50),

$$
\begin{aligned}
I^{\infty}(y;\theta^*) &= \frac{\delta}{2}\frac{\mu_\alpha(\theta^*)}{2-\delta(\mu_\alpha(\theta^*)+\mu_\beta)}y && (59)\\
&= \frac{\delta}{2}\frac{\int_{\theta\in\Theta}\mu_\alpha(\theta)\xi(\theta)\mathrm{d}\theta}{2-\delta\left(\int_{\theta\in\Theta}\mu_\alpha(\theta)\xi(\theta)\mathrm{d}\theta+\mu_\beta\right)}y, && (60)\\
&< \frac{\delta}{2}\left(\int_{\theta\in\Theta}\frac{\mu_\alpha(\theta)}{2-\delta(\mu_\alpha(\theta)+\mu_\beta)}\xi(\theta)\mathrm{d}\theta\right)y, && (61)\\
&= \frac{\delta}{2}\left(\int_{\theta\in\Theta}\frac{\frac{\mu_\alpha(\theta)}{1-\delta(\mu_\alpha(\theta)+\mu_\beta)}}{\frac{2-\delta(\mu_\alpha(\theta)+\mu_\beta)}{1-\delta(\mu_\alpha(\theta)+\mu_\beta)}}\xi(\theta)\mathrm{d}\theta\right)y, && (62)\\
&< \frac{\delta}{2}\left(\int_{\theta\in\Theta}\frac{\mu_\alpha(\theta)}{1-\delta(\mu_\alpha(\theta)+\mu_\beta)}\xi(\theta)\mathrm{d}\theta\right)\left(\int_{\theta\in\Theta}\frac{1}{\frac{2-\delta\left(\mu_\alpha(\theta)+\mu_\beta\right)}{1-\delta(\mu_\alpha(\theta)+\mu_\beta)}}\xi(\theta)\mathrm{d}\theta\right)y, && (63)\\
&< \frac{\delta}{2}\left(\frac{\int_{\theta\in\Theta}\frac{\mu_\alpha(\theta)}{1-\delta(\mu_\alpha(\theta)+\mu_\beta)}\xi(\theta)\mathrm{d}\theta}{\int_{\theta\in\Theta}\frac{2-\delta(\mu_\alpha(\theta)+\mu_\beta)}{1-\delta(\mu_\alpha(\theta)+\mu_\beta)}\xi(\theta)\mathrm{d}\theta}\right)y = I^{\infty}(y;\xi) && (64)
\end{aligned}
$$

where the first inequality comes from Jensen's inequality and the fact that $\frac{\mu_\alpha(\theta)}{2-\delta(\mu_\alpha(\theta)+\mu_\beta)}$ is convex in $\mu_\alpha(\theta)$, the second inequality comes from Holdër's inequality, and the third inequality comes from Jensen's inequality and the fact that $\left(\frac{2-\delta(\mu_\alpha(\theta)+\mu_\beta)}{1-\delta(\mu_\alpha(\theta)+\mu_\beta)}\right)^{-1}$ is concave in $\mu_\alpha(\theta)$. □

Suppose next that μ_α is independent of θ and thus known and that $\mu_\beta(\theta^*) = \int_{\theta\in\Theta}\mu_\beta(\theta)\xi(\theta)\mathrm{d}\theta$. As in the first case, learning induces more investment.

Proposition 4.4. *Suppose that μ_α is independent of θ and that $\mu_\beta(\theta^*) = \int_{\theta\in\Theta}\mu_\beta(\theta)\xi(\theta)\mathrm{d}\theta$. Then, learning increases investment, i.e.,*

$$I^{\infty}(y;\xi) > I^{\infty}(y;\theta^*). \tag{65}$$

Proof. The proof follows the same steps as the proof of Proposition 4.3. That is, from (42) and (50),

$$I^\infty(y;\theta^*) = \frac{\delta}{2}\frac{\mu_\alpha}{2-\delta(\mu_\alpha+\mu_\beta(\theta^*))}y \tag{66}$$

$$= \frac{\delta}{2}\frac{\int_{\theta\in\Theta}\mu_\alpha\xi(\theta)\mathrm{d}\theta}{2-\delta\left(\mu_\alpha+\int_{\theta\in\Theta}\mu_\beta(\theta)\xi(\theta)\mathrm{d}\theta\right)}y, \tag{67}$$

$$< \frac{\delta}{2}\left(\int_{\theta\in\Theta}\frac{\mu_\alpha}{2-\delta\left(\mu_\alpha+\mu_\beta(\theta)\right)}\xi(\theta)\mathrm{d}\theta\right)y, \tag{68}$$

$$= \frac{\delta}{2}\left(\int_{\theta\in\Theta}\frac{\frac{\mu_\alpha}{1-\delta(\mu_\alpha+\mu_\beta(\theta))}}{\frac{2-\delta\left(\mu_\alpha+\mu_\beta(\theta)\right)}{1-\delta(\mu_\alpha+\mu_\beta(\theta))}}\xi(\theta)\mathrm{d}\theta\right)y, \tag{69}$$

$$= \frac{\delta}{2}\left(\int_{\theta\in\Theta}\frac{\mu_\alpha}{1-\delta(\mu_\alpha+\mu_\beta(\theta))}\xi(\theta)\mathrm{d}\theta\right)\left(\int_{\theta\in\Theta}\frac{1}{\frac{2-\delta\left(\mu_\alpha+\mu_\beta(\theta)\right)}{1-\delta(\mu_\alpha+\mu_\beta(\theta))}}\xi(\theta)\mathrm{d}\theta\right)y, \tag{70}$$

$$< \frac{\delta}{2}\left(\frac{\int_{\theta\in\Theta}\frac{\mu_\alpha}{1-\delta(\mu_\alpha+\mu_\beta(\theta))}\xi(\theta)\mathrm{d}\theta}{\int_{\theta\in\Theta}\frac{2-\delta\left(\mu_\alpha+\mu_\beta(\theta)\right)}{1-\delta(\mu_\alpha+\mu_\beta(\theta))}\xi(\theta)\mathrm{d}\theta}\right)y = I^\infty(y;\xi). \tag{71}$$

□

While learning with unbiased beliefs unambiguously decreases consumption and increases investment, the effect on total extraction is ambiguous. To see how the ambiguity arises, consider the case in which μ_β is independent of θ and thus known. From (41) and (42), total extraction under learning is

$$2\left(C^\infty(y;\xi^*)+I^\infty(y;\xi^*)\right) = \frac{\int_{\theta\in\Theta}\frac{2-\delta(\mu_\alpha(\theta)+2\mu_\beta)}{1-\delta(\mu_\alpha(\theta)+\mu_\beta)}\xi(\theta)\mathrm{d}\theta}{\int_{\theta\in\Theta}\frac{2-\delta(\mu_\alpha(\theta)+\mu_\beta)}{1-\delta(\mu_\alpha(\theta)+\mu_\beta)}\xi(\theta)\mathrm{d}\theta}y, \tag{72}$$

while, from (49) and (50), total extraction under full information is

$$2\left(C^\infty(y;\theta^*)+I^\infty(y;\theta^*)\right) = \frac{2-\delta(\mu_\alpha(\theta^*)+2\mu_\beta(\theta^*))}{2-\delta(\mu_\alpha(\theta^*)+\mu_\beta(\theta^*))}y. \tag{73}$$

Suppose now that $\mu_\beta(\theta)$ is independent of θ and that $\mu_\alpha(\theta^*) = \int_{\theta\in\Theta}\mu_\alpha(\theta)\xi(\theta)\mathrm{d}\theta$.

Then, observe first that, by Jensen's inequality,

$$
\begin{aligned}
2\left(C^{\infty}(y;\theta^*)+I^{\infty}(y;\theta^*)\right) &= \frac{2-\delta(\mu_\alpha(\theta^*)+2\mu_\beta)}{2-\delta(\mu_\alpha(\theta^*)+\mu_\beta)}y, \\
&= \frac{\int_{\theta\in\Theta}\left(2-\delta(\mu_\alpha(\theta)+2\mu_\beta)\right)\xi(\theta)\mathrm{d}\theta}{\int_{\theta\in\Theta}\left(2-\delta(\mu_\alpha(\theta)+\mu_\beta)\right)\xi(\theta)\mathrm{d}\theta}y, \\
&> \int_{\theta\in\Theta}\frac{2-\delta(\mu_\alpha(\theta)+2\mu_\beta)}{2-\delta(\mu_\alpha(\theta)+\mu_\beta)}\xi(\theta)\mathrm{d}\theta y, \\
&= \int_{\theta\in\Theta}\frac{\frac{2-\delta(\mu_\alpha(\theta)+2\mu_\beta)}{1-\delta(\mu_\alpha(\theta)+\mu_\beta)}}{\frac{2-\delta(\mu_\alpha(\theta)+\mu_\beta)}{1-\delta(\mu_\alpha(\theta)+\mu_\beta)}}\xi(\theta)\mathrm{d}\theta y,
\end{aligned}
\tag{74}
$$

since $\frac{2-\delta(\mu_\alpha(\theta)+2\mu_\beta)}{2-\delta(\mu_\alpha(\theta)+\mu_\beta)}$ is concave in $\mu_\alpha(\theta)$. Next, observe that, by Holder's inequality,

$$
\begin{aligned}
\int_{\theta\in\Theta}\frac{\frac{2-\delta(\mu_\alpha(\theta)+2\mu_\beta)}{1-\delta(\mu_\alpha(\theta)+\mu_\beta)}}{\frac{2-\delta(\mu_\alpha(\theta)+\mu_\beta)}{1-\delta(\mu_\alpha(\theta)+\mu_\beta)}}\xi(\theta)\mathrm{d}\theta y &< \left(\int_{\theta\in\Theta}\frac{2-\delta(\mu_\alpha(\theta)+2\mu_\beta)}{1-\delta(\mu_\alpha(\theta)+\mu_\beta)}\xi(\theta)\mathrm{d}\theta\right) \\
&\quad\cdot\left(\int_{\theta\in\Theta}\frac{1}{\frac{2-\delta(\mu_\alpha(\theta)+\mu_\beta)}{1-\delta(\mu_\alpha(\theta)+\mu_\beta)}}\xi(\theta)\mathrm{d}\theta\right)y, \\
&< \frac{\int_{\theta\in\Theta}\frac{2-\delta(\mu_\alpha(\theta)+2\mu_\beta)}{1-\delta(\mu_\alpha(\theta)+\mu_\beta)}\xi(\theta)\mathrm{d}\theta}{\int_{\theta\in\Theta}\frac{2-\delta(\mu_\alpha(\theta)+\mu_\beta)}{1-\delta(\mu_\alpha(\theta)+\mu_\beta)}\xi(\theta)\mathrm{d}\theta}y \\
&= 2\left(C^{\infty}(y;\xi^*)+I^{\infty}(y;\xi^*)\right).
\end{aligned}
\tag{75}
$$

Statements in expressions (74) and (75) cannot be reconciled, which illustrates the ambiguity of the effect of learning on total extraction.

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