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A Reconsideration of Arrow-Lind: Risk Aversion, Risk Sharing, and Agent Choice

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Abstract:
We consider the original Arrow-Lind framework in which a government undertakes a risky project to be shared among many taxpayers. In our model, the taxpayers decide the level of participation in the risky project. Moreover, the amount of taxes collected by the government fully finances the public project. In this case, we show that projects cannot be evaluated only on the basis of expected benefits since the resulting tax determined by the model is incompatible with any risk sharing.

Keywords: Arrow-Lind Theorem, Risk aversion, Risk sharing, Choice

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1 Introduction

Virtually all projects in an economy yield uncertain benefits. Uncertainty is present in both public investment undertaken by governments, as well as in market activities in the private sector. In each situation, agents must share the risk and therefore must decide how much of the risk to share. Moreover, due to risk aversion, each participant faces a cost of bearing risk. Thus, each participant must choose whether and how much they are willing to contribute to these risky projects (i.e., to share the risk). It is important to understand the basis for evaluating, and thus allocating, these risky projects.

Arrow and Lind (1970) provides an early analysis of evaluating risky projects in the context of a large population of participants sharing the risk of a public investment. The Arrow-Lind theorem states that as the population of participants tends to infinity, social risk becomes negligible. More specifically, as the number of participants grows, the risk premium of a risk-averse participant (corresponding to a share of the public risky project) decreases, and, in the limit, goes to zero. Moreover, the social risk premium also goes to zero. Hence, with a large population, it is argued that projects can be evaluated on the basis of expected (net) benefits alone without any concern for risk.

The reasoning in Arrow and Lind (1970) is not based on a model in which the link between a zero risk premium and risk-neutral behavior interact, but depends on risk-averse agents who are not engaged explicitly in trading and decision-making. In this paper, we investigate the effect of letting the number of participants get infinitely large by modeling the decision mechanism of these participants. Indeed, there are choices to be modeled and decisions to be made, as well as an equilibrium that must be taken into account, before understanding the effect of the number of agents getting large as well as in the limit. In particular, in a free-market economy, the interaction of the choice of risk and the decisions by risk-averse agents determines the amount and the allocation of risk in the economy.

We first consider the original Arrow-Lind framework in which a government undertakes a risky project to be shared among many taxpayers. In our model, the taxpayers decide the level of participation in the risky project. Moreover, the amount of taxes collected by the government fully finances the public project. In this case, we show that projects cannot be evaluated only on the basis of expected benefits since the resulting tax determined by the model is incompatible with any risk sharing. That is, if the effect of the cost of bearing risk is excluded, then taxpayers refuse to contribute any amount to finance public investment. Thus, no equilibrium would ensue. Because the Arrow-Lind theorem has had a profound impact beyond the sphere of public investment under uncertainty, we then discuss the issue of risk sharing in a more market oriented setting, with trading.

2 Choosing and Sharing a Public Investment

In this section, we extend the Arrow-Lind framework by allowing the taxpayers to choose their levels of contribution to the public investment and by endogenizing the total tax such that the project is fully funded. The inclusion of decision making in the Arrow-Lind framework has a profound effect on the interpretation of limiting cases. Specifically, we shows that perfect risk spreading (due to the population of taxpayers tending to infinity) does not imply that social risk is negligible. In other words, if the public investment is evaluated on the basis of expected benefit, then risk-averse taxpayers are unwilling to contribute. We first present the model and characterize the equilibrium for any finite number of taxpayers. We then discuss the limiting case. This model is in the spirit of the model used in Arrow and Lind (1970). It does not consider the free rider problem, i.e., the revelation of truthful valuations for the taxpayers. Prices are uniform and the government's budget is balanced. This model is used primarily to study the effect of risk aversion and risk sharing.

Consider an economy with a government and $n > 0$ taxpayers. The government undertakes a public investment. The random benefit for the project is represented by $\tilde{B} = \overline{B} + \tilde{\varepsilon}$ where $\overline{B}$ is the expected social benefit and $\tilde{\varepsilon}$ is a random shock.[1] Taxpayer i's disposable income is

$$\tilde{Y}_i = A_i + \tilde{B}s_i - ps_i, \tag{1}$$

where A_i is income excluding the benefits emanating from the public investment,[2] and $(\tilde{B}-p)s_i$ is taxpayer i's random return for the public investment. Here, $s_i \in [0,1]$ is taxpayer i's level of participation in the public investment. Hence, given s_i, $\tilde{B}s_i$ is the random benefit accruing to taxpayer i whereas ps_i is the contribution toward the total tax raised by the government, $p > 0$.[3] At the aggregate level, taxpayers must fully contribute to the public investment, i.e., $\sum_{i=1}^{n} s_i = 1$. Unlike Arrow and Lind (1970), the disposable income is explicitly written as the difference between income $A_i + \tilde{B}s_i$ and taxes ps_i.[4]

To simplify the analysis, we assume that the taxpayers exhibit constant absolute risk aversion over disposable income and the random benefits of the public investment are normally distributed. That is, the public project is potentially harmful. However, this does not detract from the results since all decisions variables are positive.

Assumption 2.1. *The coefficient of absolute risk aversion is $a > 0$ for any taxpayer. In other words, the utility function for wealth Y_i is exponential: $u(Y_i) = -e^{-aY_i}$.*

Assumption 2.2. *$\tilde{\varepsilon} \sim N(0, \sigma^2)$.*

Assumptions 2.1 and 2.2 yield a closed-form solution of the certainty equivalent. Given the choice s_i and the total tax p, taxpayer i's certainty equivalent is[5]

$$CE(s_i, p) = A_i + (\overline{B} - p)s_i - a\sigma^2 s_i^2/2 \tag{2}$$

[1] A tilde distinguishes a random variable from its realization.

[2] Unlike Arrow and Lind (1970), we assume that private income is nonstochastic. This simplification has no bearing on the analysis.

[3] Foldes and Rees (1977) considers the role of the fiscal system and public expenditures in the Arrow-Lind framework.

[4] We implicitly assume that A_i includes transfers from the government which are unrelated to agent i's tax contributions.

[5] The certainty equivalent is implicitly defined by $\mathbb{E}u(A_i + (\tilde{B} - p)s_i) = u(CE(s_i, p))$.

where $A_i + (\overline{B} - p)s_i$ is the expected disposable income and $a\sigma^2 s_i^2/2$ is the risk premium. Following the notation in Arrow and Lind (1970), let

$$k(s_i) \equiv a\sigma^2 s_i^2/2 \tag{3}$$

be the risk premium so that $\sum_{i=1}^{n} k(s_i)$ is the social risk premium.

Having presented the setup, we now define the equilibrium. In equilibrium, given the total tax raised by the government, each taxpayer chooses an optimal level of contribution. Moreover, the value of the total tax induces full participation on the part of the taxpayers. In other words, p^* clears the market so that the public investment undertaken by the government is desired by the taxpayers. These two aspects, not present in Arrow and Lind (1970), reflect the idea that in a functioning democracy, public investment is not imposed by the government, but requires the agreement of voting taxpayers. Specifically, in our model, the taxpayers express their willingness to take part in the project by *choosing* their level of contribution whereas in Arrow and Lind (1970), the level of contribution is set *ex ante*, i.e., $s_i = 1/n$. Moreover, in our model, the total tax is endogenous, which reflects the *market* value of the project. In the analysis, the equilibrium value of the total tax forms the basis for evaluating a public investment.

Definition 2.3. *The tuple* $\{\{s_i^*(p^*)\}_{i=1}^{n}, p^*\}$ *is an equilibrium if*

1. *Using* (2)*, given* $p^* > 0$*, for* $i = 1, \ldots, n$*,*

$$s_i^*(p^*) = \arg\max_{s_i} \{A_i + (\overline{B} - p^*)s_i - a\sigma^2 s_i^2/2\}. \tag{4}$$

2. *Given* $\{s_i^*(p^*)\}_{i=1}^{n}$*,* p^* *satisfies* $\sum_{i=1}^{n} s_i^*(p^*) = 1$*.*

Proposition 2.4 characterizes the unique equilibrium for a finite number of taxpayers. For any finite number of taxpayers, the project is fully funded and always desired by the taxpayers.

Proposition 2.4. *For $n < \infty$, there exists a unique equilibrium. In equilibrium, each taxpayer contributes a fraction*

$$s_i^*(p^*) = 1/n \tag{5}$$

to the public project and the tax raised by the government is

$$p^* = \overline{B} - \frac{a\sigma^2}{n}. \tag{6}$$

Proof. The first-order condition corresponding to (4) is $\overline{B} - p - a\sigma^2 s_i = 0$ which yields $s_i^*(p) = \frac{\overline{B}-p}{a\sigma^2}$. Plugging $s_i^*(p)$ into $\sum_{i=1}^n s_i^*(p^*) = 1$ and solving for p^* yields (6). Plugging (6) back into $s_i^*(p) = \frac{\overline{B}-p}{a\sigma^2}$ yields (5). □

From Proposition 2.4, for any finite number of taxpayers, $s_i^*(p^*) = 1/n$ as in Arrow and Lind (1970). Moreover, in this case, both the risk premium and the social risk premium tend to zero as the population of taxpayers tends to infinity. That is, plugging $s_i^*(p^*) = 1/n$ into (3) yields the equilibrium risk premium

$$k^*(s_i^*(p^*)) = \frac{a\sigma^2}{2n^2}. \tag{7}$$

Using (7), $\lim_{n\to\infty} k(s_i^*(p^*)) = \lim_{n\to\infty} n \cdot k(s_i^*(p^*)) = 0$. The reason is that *given the demand schedule of the taxpayer derived under a finite number of taxpayers*, the gamble disappears in the limit. Consequently, the value of the public investment (in terms of the total tax) tends to the expected benefit, i.e., from (6), $\lim_{n\to\infty} p^* = \overline{B}$.

Note that a vanishing (social) risk premium does not imply that projects can be evaluated on the basis of expected benefit alone. The issue here is that any evaluation that disregards risk implies a profound change in the optimal behavior of risk-averse taxpayers, i.e., there is a discontinuity in the equilibrium in the limit. To see this, suppose that a vanishing social risk premium is interpreted as making social cost negligible so that the value of the project depends only on the expected benefit. Hence, the total tax is set equal to the expected benefit as in the limiting case, i.e., $p^* = \overline{B}$, which removes any consideration for uncertainty and risk aversion in evaluating the risky public project. Now, regardless of the number of taxpayers, each taxpayer makes an optimal decision on whether and how much to contribute to the project by taking account of the total tax. In this case, each taxpayer receives a zero expected return, i.e., $\overline{B} - p^* = 0$. With zero expected return, no risk-averse taxpayer has an incentive to contribute to the risky project. Formally, from (2), for $s_i \in [0, 1]$,

$$\left.\frac{\partial CE(s_i, p^*)}{\partial s_i}\right|_{p^*=\overline{B}} < 0. \tag{8}$$

That is, there is no gain from contributing on the part of the taxpayers, which induces each taxpayer to opt out of the public investment. There is thus a discontinuity in the sense that as long as the project is not evaluated on the basis of expected return, the taxpayers are willing to fully contribute to the project, but as soon as it is evaluated on the basis of expected return, the taxpayers are unwilling to participate. This discontinuity implies that the expected value of benefits does not closely approximate the correct measure of benefits defined in terms of willingness to pay for an asset with an uncertain return when n is large since the limiting case is not an equilibrium itself. Formally,

Proposition 2.5. *Suppose that the project is evaluated on the basis of expected payoff, i.e.,* $p^* = \overline{B}$*. Then, there is no equilibrium since for all* i*,* $s_i^*(p^*)\big|_{p^*=\overline{B}} = 0$.

3 Risk Sharing in Markets

The previous discussion focuses on the evaluation of public investment under uncertainty. We now discuss a more general framework in which several agents share risky projects via the financial sector. This is relevant since the financial sector of an economy plays an increasingly important role in a growing and more complex economy that generates more savings and investments. The financial sector influences not only the various types of risk undertaken in the economy, but also how these risks are shared among agents.

One standard approach to study the financial sector is to assume that agents are risk-averse. Thus, the Arrow-Lind argument can be exported to the private financial sector. That is, with a very large number of risk-averse investors, risk spreading implies that the risk premium goes to zero eliminating any exposure to, and concern for risk. In other words, while the shareholders are risk-averse, there is no need to account for their risk aversion, i.e., they act as if they are risk-neutral. The implication that a zero risk premium in the limit implies risk neutrality leads falsely to the conclusion that it is equivalent to assume that agents are risk neutral and not risk-averse. However, no matter what the conclusion of agents' behavior in a model in which the number of agents is large (so that the exposure to risk is very small), the results depend on the assumption that agents are risk averse. Indeed, agents acting as if they are risk neutral in the limit is a conclusion of the model in which agents are risk-averse and all risk disappears. However, the conclusion of the model with respect to behavior should not lead to a change in the assumptions of the model, i.e., that agents are not risk-averse.

To see this, consider several risk-averse investors who must decide whether to invest in a risky firm owned by a risk-averse entrepreneur. Specifically, consider an economy with one entrepreneur and several investors whose objective is to maximize the expected utility of wealth over a portfolio of a risky asset and a risk-free asset. The entrepreneur, the founder and initial owner of the firm, issues equity shares that are claims on the profit generated by the firm. Investors do not have entrepreneurial prospects. However, they do have initial wealth that they use to purchase shares of the risky and the risk-free asset.

As in the case of sharing risk of a public investment, it can be shown that there is no equilibrium with perfect risk spreading (i.e., when the number of investors goes to infinity) because no risk averse agent wishes to trade. While risk-neutral investors have no concern for risk and are indifferent to sharing risk, risk-averse agents in an economy with perfect risk spreading do not want to share risk. To understand this result, consider a group of risk-averse investors deciding whether to buy shares from a risk-averse entrepreneur. The market price of the risky asset is endogenous and is affected by the number of agents. On the one hand, when the number of investors grows, risk spreading increases the price of the risky asset and, in the limit, is equal to the expected payoff of the asset, so that the investors, who are risk-averse, get no risk premium and, thus, do not want to hold any of the random asset. In other words, perfect risk spreading implies a price equal to the expected payoff so that there is no gain in engaging in risk sharing, and, in addition, holding even a tiny share of the risky asset makes it costly in terms of risk. On the other hand, with the price of the asset equal to its expected value, the entrepreneur would like to push the entire investment off on the investors. This is not viable since the risk averse agents refuse to accept the trade, and, thus, no trading is possible.

This result is in stark contrast to the outcome derived from assuming risk-neutral investors. If investors are risk neutral, the risk-averse entrepreneur passes all the risk to the investors. Hence, the assumption of risk aversion and the explicit modeling of trade have a powerful effect on the conclusion of the model. Indeed, the limiting price of the asset is such that expected return is zero and any risk-averse investor still cares about risk and incurs a cost. The reason for this difference is due solely to whether trading is modeled. It should be pointed out that, in the limit, even when trading is not modeled, the agent does not become risk-neutral, but the risk premium goes to zero.[6] On the other hand, when trading is modeled, in the limit, risk-averse investors instead of behaving in a risk neutral way, do not, in fact, share risk.

[6]This is true since the gamble disappears and so does the risk. Consider N agents who are forced to share a risk. Dividing a given risk more and more finely causes the risk premium to vanish not because the agents become risk-neutral but because the gamble disappears.

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