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Risk-Sharing Networks

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Abstract:
This paper considers the formation of risk-sharing networks. Following empirical findings, we build a model where risk-sharing takes place between pairs of individuals. We ask what structures emerge when pairs can agree to form links, but people cannot coordinate links across a population. We consider a benchmark model where identical individuals commit to share their monetary holdings equally with linked partners. We compare efficient networks to equilibrium networks. Efficient networks can (indirectly) connect all individuals and involve full insurance. However, equilibrium networks connect fewer individuals. There is an externality : when breaking a link individuals do not take into account the negative effect on others distant in the network. The network formation process can lead identical individuals to be in different positions and thus have different risk-sharing outcomes. These results may help explain empirical findings that risk-sharing is often not symmetric or complete.

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I. Introduction

In this paper, we study the formation of risk sharing networks. In many settings, formal insurance mechanisms are not available. People often mitigate risk by making insurance arrangements among themselves. We see people sharing income and helping each other out in many different countries and settings. There is now a large body of empirical work on risk-sharing arrangements, and one major finding of this research is that informal risk-sharing is often not complete within the observed set of individuals.[1] That is, within a village, people do not enjoy the benefits of complete risk-sharing across individuals in the village. One reason, researchers suspect, is that risk-sharing does not take place at the village level, but within families and between individuals. In their study of the rural Phillipines, Fafchamps and Lund (2003, p. 216) find, for example, that "mutual insurance does not appear to take place at the village level; rather households receive help primarily through networks of friends and relatives." [2]

In line with these empirical findings, we build a model where risk-sharing takes place between pairs of individuals. Most theoretical work on informal insurance has assumed that risk-sharing takes place within groups, where the group can be as large as the village or as small as two people.[3] A notable recent exception is Bloch, Genicot, and Ray (2004) which we discuss below. We follow the empirical research and suppose that within a given population, there is no cohesive risk-sharing group, per se. Rather, individuals can form bilateral risk-sharing relations. And individuals can have many such bilateral relationships, but there is no requirement that if one person shares income with a second and the second shares with a third that the first and the third automatically have a relationship with each other as well. That is, there is a network of sharing relationships. In our model, we assume that people make bilateral transfers and can make transfers to each other *only if* they have previously established a relationship that allows them to observe income levels and commit to a sharing agreement. It is costly to form such a relationship, as there may be investments that allow monitoring or the ability to enforce a

[1] For seminal contributions see Townsend (1994) and Udry (1996).

[2] These findings support suspicions of earlier work that finds incomplete insurance at the village level. For example, Townsend (1994, p. 541) writes that "kinship groups or networks among family and friends might provide a good, if not better, basis for testing the risk-sharing theory." In recent work, De Weerdt (2004) and Dercon and De Weerdt (2005) analyze a survey of villagers in Tanzania asking, "Can you give a list of people from inside or outside of Nyakatoke, who you can personally rely on for help and/or that can rely on you for help in cash, kind, or labour?" They notably find that, on average, households are linked to relatively few other households.

[3] See, for example, Kimball (1988), Coate and Ravaillion (1993), and Ligon, Thomas, and Worrall (2002).

risk-sharing agreement.[4] Our key assumptions, then, are that relationships are bilateral and establishing such a relation is costly.

We study and contrast the equilibrium and the efficient patterns of risk-sharing relations when individuals form bilateral agreements. We consider the formation of risk-sharing links. We ask what structures will emerge when pairs can agree to form links but agents cannot coordinate link formation across the whole population. That is, we consider a decentralized process of link formation. There is a two stage game. First, agents form links. Second, income shocks are realized, and agents earn utility from after sharing with their bilateral relations.

We consider a benchmark model where identical individuals commit to share their monetary holdings equally with their linked partners. We show that if individuals are committed to share income equally within pairs and interact repeatedly with their neighbors, they can end up sharing income equally within components of the network. The process is useful for our analysis for two reasons. This outcome corresponds to the highest level of insurance that can be secured in a network. Second, it matches the efficient level of insurance. Thus any divergency between efficient networks and equilibrium networks would come from the network formation process. That is, we hold the risk-sharing level as constant and ask how the necessity of forming relations bilaterally affects the configuration of the risk-sharing arrangements.

In this setting, we have several findings.

First, efficient risk-sharing networks can (indirectly) connect all individuals within a society and involve full insurance. That is, efficient networks can result in the equivalent of full-income pooling with a population, despite bilateral relations and commitment costs.

Second, equilibrium risk-sharing networks, in general, connect fewer individuals than efficient risk-sharing networks. This inefficiency arises because of a basic externality in our risk-sharing networks. While income transfers are bilateral, individuals can benefit from their partners' risk-sharing relations with others, and those parties relations with yet others, and so on. When forming a single relation - or, critically, breaking a relation - individuals do not take into account the effect on others distant in the network. Hence, models of risk-sharing with commitment in groups would overstate the benefits that arise from informal risk-sharing.

[4] For example, an investment could be arranging a marriage with a member of another household. This marriage would allow both greater observability and greater ability to punish (say, socially) the partner for not sharing income. We discuss this further below.

Third, we find that equilibrium networks divide the population into sets of different sizes and, hence, individuals are in asymmetric positions. That is, individuals have different risk-sharing outcomes, despite that individuals have identical preferences and their incomes are identically distributed. In general, we show that equilibrium risk-sharing networks involve separate components - with one smaller than the others. A single component cannot be too large, because the benefits of linking to an individual in a larger and larger component eventually will not exceed the cost. Hence, the largest component is of bounded size. A second component is smaller. If the second component is too large, then there is an incentive for a pair to make a link between components. But such a super-sized component is itself not stable. Agents on the edges would be cut out. This finding of asymmetric outcomes suggest a more precise reason for the empirical finding of incomplete risk-sharing. The process of network formation can lead similar individuals to be in different positions, and thus have different outcomes.

Fourth, as hinted at above, there is an inherent instability in risk-sharing networks. Components must be neither too large nor too small, to prevent incentives to cut links or form links between components. Often, the population cannot be divided into such components, and there is no pairwise stable network.

Finally, in an environment where agents continually break and make links, agents who find themselves on the edge of a network are the most vulnerable to being cut out completely. They bring little benefit because they have connections to fewer other agents. In a dynamic model of network formation we show that, if pairwise stable networks exists, there is eventual convergence to a pairwise stable network. When there is no pairwise stable network, we see cycles - where the size of connected components grow and shrink over time, and agents at the edge of networks are cut off. This finding leads us to question the idea of stable risk-sharing networks. In a snap-shot, the risk-sharing relations we observe may just be part of a long cycle that changes over time.

Our paper focuses on the formation of risk-sharing relationships. Previous theoretical literature on informal insurance focuses, for the most part, on the enforcement of risk-sharing agreements. Typically, there is a repeated game where a given set of agents receive income shocks in each period. They are then supposed to share income with another given set of agents. If an agent does not share income, he is punished in the future by exclusion from the income-sharing group.[5] The question then becomes how the severity of the punishment determines the level of risk-sharing that can be sustained in equilibrium. Bloch, Genicot, Ray's recent (2004) paper considers this enforcement problem when income is shared in pairs. Income is transferred only between pairs of agents, where many pairs can transfer income to one another. The question then becomes what pattern of transfers is sustainable in an equilibrium of a repeated game. That is, they define a risk-sharing network as the pattern of equilibrium transfers. We define a risk-sharing network differently in our present effort. A risk-sharing network is a pattern of existing relations where agents can commit to sharing income. There is no enforcement problem per se.

An illustration might clarify the distinction and our contribution. Consider a population and suppose that risk sharing takes place within extended families. Our paper asks how a family might form a relation with another family - by marriage say - in order to establish a risk-sharing relation. (There is evidence in India, for example, that marriages of daughters are arranged to maximize gains from risk sharing (Rosenzweig and Stark (1989)). Establishing this relation is costly, involving a dowry and marriage ceremony etc., and the relation commits the parties to future income sharing, say, due to a social norm or social punishment in case of non-sharing. The marriage pattern would then be our network. In contrast, for Bloch, Genicot, Ray (2004), there is no ex ante formation of relations and transfers are not restricted by marriage. Rather any individual can share with any other individual in the population. But sharing must be enforced through the threat of withdrawing future interaction. They determine equilibrium patterns of bilateral transfers, and use the term network to describe these patterns.

Our paper, then, contributes to the growing theoretical literature on the formation of social networks. We develop the first model of ex ante formation of links that are later used for risk pooling. We apply the equilibrium notion of pairwise stability, introduced by Jackson and Wolinsky (1996), to a context of risk-sharing, and we characterize pairwise stable and efficient networks. Our model is representative of a general setting: links have positive externalities, individual benefits depend only on the size of components, and individuals always benefit (at a decreasing rate) from an increase in the number of people in their component. Our results would hold in any economic environment that yields these network characteristics.

More generally, our paper considers the relationship between individual interactions and ag-

[5] In a bilateral situation, this "group" would consist of the one other person in the pair. In a larger population, the deviating agent could be punished by all other agents or some subset of agents (see Genicot and Ray (2003)).

gregate outcomes.[6] In the present paper, individual interactions determine the risk-sharing networks, which in turn determines the extent of risk-sharing within and across the population. As discussed above we have a striking finding. Ex ante symmetric agents can end up with very different risk-sharing outcomes. That is, different outcomes across individuals may not be the result of some underlying heterogeneity but the result of the interactions among agents.

The rest of the paper is organized as follows. In Section II we introduce concepts from the theory of networks to describe the pattern of risk-sharing relations and develop our model of bilateral risk-sharing. In Section III we solve for efficient networks. In Section IV we formulate a two stage network formation game: in the first stage agents form bilateral relations, in the second stage they share incomes with these relations. We solve for the pairwise stable networks, networks where no more pairs have an incentive to form a relation, and no individual has incentive to break a relation. We then compare pairwise stable networks to efficient networks. In section V, we look at a dynamic model of network formation, where agents may form and break relations over time. We conclude in Section VI.

II. The Model

Consider a society of n individuals. Individuals are risk-averse and face shocks to their incomes. Each individual's income, y_i, is a random variable, and incomes are independent and identically distributed with mean $\overline{y}$ and variance σ^2. People have identical preferences and we represent their utility by a utility function v, which is increasing and strictly concave. Formal insurance mechanisms are not available. The only way for people to mitigate risk is to make insurance arrangements among themselves. We assume that two individuals can make transfers to each other *only if* they had previously established a risk-sharing relation. Our key assumptions is that individuals can only make bilateral transfer and establishing a bilateral relation is costly.

A. Definition of Links and Networks

To model bilateral risk-sharing in a large population, we use tools from the theory of networks. An individual i and j can form a risk-sharing relationship by each incurring a cost c.[7] If they incur

[6] We see this theme in, for example, Kirman (1993) and the volume Kirman and Zimmermann (2001).

[7] We can think of this cost in utility terms. Agents then have additively separable utility functions, as described below.

this cost, we say they have a *link*. We think of the link cost as a fixed cost that must be incurred by each of the agents. That is, one agent cannot compensate another agent for the expense of building a link. This assumption reflects the idea that some costs, e.g., the time incurred to build a relation, are not easy to compensate or transfer.[8] In the case of marriage, this cost would be, for example, the time and money involved in courtship and providing a dowry. After the link is formed, the cost is sunk.

We represent links and a network of links with the following notation: g is an $n \times n$ matrix, where $g_{ij} = 1$ when i and j have a link, i.e., have established a risk-sharing relation, and $g_{ij} = 0$ otherwise. We assume that risk-sharing relations are mutual, so that $g_{ij} = g_{ji}$. By convention, $g_{ii} = 0$. We say there is a *path* between two individuals i and j in the graph g if there exists a sequence of individuals $i_1, ..., i_k$ such that $g_{ii_1} = g_{i_1 i_2} = ... = g_{i_k j} = 1$. A subset of individuals is *connected* if there is a path between any two individuals in the subset. A *component* of the graph g is a maximal connected subset. Components provide a partition of the population. A graph is *minimally connected* when the removal of any link increases the number of components.

In what follows, we first model, for a given network g, how much each individual gains in risk reduction. This process results in individual benefit functions that depends on the network. We then use these benefit functions to solve for efficient networks and specify and analyze our network formation game.

B. Risk Sharing in Networks

Given a network g, how do people share risk?

We consider a benchmark model where individuals commit to share their monetary holdings equally with their linked partners. Monetary holdings are individual incomes net any transfers. Specifically, suppose that people meet repeatedly over time after the income shock. For any two linked individuals, they are committed to share their monetary holdings equally *every time they meet*. For example, suppose we are studying an agricultural setting and at the end of the growing

[8] In an alternative model, agents could compensate each other for link costs. Indeed, we could view link costs as another income shock to be shared between agents in a risk-sharing relation. Such a model would still lead to a divergence between individual, or pairwise, incentives to form links and the criteria of a social planner. A pair of agents would only consider the impact on their own utility, not on the utility of everyone in the population. The difference we would see is the well-known divergence between the effect of decision on the marginal agent (or pair of agents) and the average agent.

season, farmers' incomes are realized. They then visit with their linked partners, and share incomes. Then visit linked relations again, and so on, until the beginning of the following season. This process is a benchmark for income sharing, since - as we show below - when pairs meet often enough it achieves complete risk-sharing within a component of a network. It is as if individuals in the component completely pooled their income. By sharing net monetary holdings in bilateral relations, individuals mitigate risk throughout the connected network. Complete income pooling within a set of individuals yields the highest possible aggregate utility.

To illustrate consider a population of three people, $n = 3$. Agent 1 is linked to agent 2, and agent 2 is linked to agent 3, but agent 1 is not linked to agent 3. That is, the network is a star. Agents' receive incomes y_1, y_2, and y_3. The agents then interact as follows. Agents randomly meet their linked partners such that each agent meets each of his linked partners at least once. Whenever agents meet they share equally their current monetary holdings. For example, first agent 1 meets agent 2, and they share incomes. Each then has $\frac{y_1+y_2}{2}$. Agent 3 then meets agent 2, and they share their current monetary holdings. Agent 2 and agent 3 then each have $\frac{\frac{y_1+y_2}{2}+y_3}{2}$. Agent 2 then meets agent 1. They share monetary holdings equally, and agent 1 and agent 2 then have $\frac{\frac{y_1+y_2}{2}+\frac{\frac{y_1+y_2}{2}+y_3}{2}}{2} = \frac{y_1+y_2+y_3}{4} + \frac{y_1+y_2}{8}$ each. Agent 3 still has $\frac{\frac{y_1+y_2}{2}+y_3}{2} = \frac{y_1+y_2}{4} + \frac{y_3}{2}$. And so on.

We formalize such random meetings as follows. There are T rounds of interactions. (1) Every pair of linked individuals meets at least once in each round. (2) An individual interacts with only one of his links at any one time.[9] This interaction process represents risk-sharing following a realization of income. Again, we emphasize that it is a benchmark. It implicitly assumes that agent's cannot hide transfers from others nor exaggerate transfers to others.

With this interaction process we can show the following result. As the number of rounds of interaction approaches infinity, monetary holdings will be equalized for all individuals belonging to the same component of the network. All formal proofs are given in Appendix.

Proposition 1. *As the number of rounds increases, (at $T \to \infty$), an individual's money holdings converge to the mean level of the income shocks in his risk-sharing component.*

[9] A pair of linked individuals may meet more than once in a round, but the number of times they meet is bounded. Also, an individual can (and will) interact with different neighbors in one round, but these interactions will take place at different times.

To prove this result, we first show that the dispersion of monetary holdings within components must decrease weakly after each round. We then show that the decrease is strict when all individual monetary holdings are not equal. We finally show that dispersion must converge to zero as the number of rounds becomes arbitrarily large.

This result shows that if individuals are committed to share income equally within pairs and interact repeatedly with their neighbors, they end up sharing income equally within components. The process has two noteworthy features. First, individuals do not need to have information on the past transfers between people. They only need to know the current level of monetary holdings of their neighbors when they meet them. Second, modifying the interaction process could provide one way to introduce frictions in the way transfers flow through links. For instance, if the process takes place over a finite number of rounds, monetary holdings within the component will, in general, not equalize within the component, but depend on an agent's position within the component. Or if a critical pair is left out of the interaction process, individual incomes may converge to different values.[10]

This outcome corresponds to the highest level of insurance that can be secured, given the constraint that income-sharing occurs between pairs (e.g., not in groups).[11] It thus allows us to study how the formation of a network affects outcomes, where the best level of risk-sharing in a network is the same as the best level possible. That is, we hold the risk-sharing level as constant and ask how the necessity of forming relations bilaterally affects the configuration of the risk-sharing arrangements.

With this interaction process, risk-sharing benefits only depends on the number of individuals in the risk-sharing component. If s denotes the size of this component, the expected utility $u(s)$ is given by

$$u(s) = Ev\left(\frac{y_1 + ... + y_s}{s}\right)$$

where the expectation is taken over all realizations of incomes for all individuals $(1, \ldots, s)$ in the component. Thus, $u(1) = Ev(y_1)$ is the expected utility for an individual who has no links. Let $s_i(g)$ denote the size of i's component in g. The expected utility of individual i from belonging to

[10] There are other ways to understand equal sharing. Bloch et al. (2004) show that equal sharing within a pair is one fixed point of bilateral transfers and focus on equal sharing because it is a "social norm."

[11] More precisely, under the constraint that only linked individuals can make transfers, complete insurance within components maximizes the sum of ex-ante utilities over all possible transfers.

network g is $u(s_i(g))$. This reduced-form expected utility function u captures all the properties of the distribution of income shocks, the primitive utility function v, and the graph that matter for our analysis of efficient networks and our network formation game.

In general, the expected utility function u satisfies two properties. With v strictly increasing and concave, expected utility $u(s) = Ev\left(\frac{y_1+...+y_s}{s}\right)$ is non-decreasing and bounded from above (i.e., $u(s) = Ev\left(\frac{y_1+...+y_s}{s}\right) \leq v(\overline{y})$ (see Rothschild & Stiglitz (1971)). We make the following additional assumption - we assume that $u(s)$ is increasing in the size of the component *at a decreasing rate.*

Assumption: $\forall s, u(s+1) > u(s)$ and $u(s+2) - u(s+1) < u(s+1) - u(s)$.

This "concavity" assumption is satisfied, for instance, when the primitive utility function, v, is quadratic, as shown in the example below.[12]

Example 1. *Consider a quadratic utility function $v(y) = y - \lambda y^2$, where λ is a positive parameter such that $\lambda < \frac{1}{2y}$ for all values of y. Observe that a larger λ corresponds to a more risk-averse individual. By the law of large numbers, $u(s)_{s\to\infty} = v(\overline{y})$. It is then easy to see that $u(s) = v(\overline{y}) - \frac{\lambda\sigma^2}{s}$, where recall $\overline{y}$ is the mean of a distribution of income and σ^2 is the variance. With this utility function, $u(s)$ is greater when risk-aversion, measured by λ, is lower. We can see that $u(s)$ is increasing in s, and $u(s)$ satisfies our "concavity" assumption.*

III. Efficient Networks

We now characterize efficient networks. We find that two cases emerge depending on the properties of aggregate expected utility. In one case, where aggregate expected utility function is convex, the benefits of adding more people to a component gets higher and higher. The efficient network then either contains one component connecting the whole population or involves no links. In the other case, where aggregate expected utility is concave, the benefits of adding more people to a component gets smaller and smaller. The efficient network involves intermediate size components that balances benefits and costs. We later compare pairwise stable networks to efficient networks. We find that pairwise stable networks are always underconnected from a social welfare perspective. This result expresses a positive externality inherent to risk-sharing in networks.

[12] In general, since u is non-decreasing and bounded, the number of values of s such that $u(s+2) - u(s+1) \leq u(s+1) - u(s)$ is infinite.

In order to study efficiency issues, we introduce a simple welfare measure. Let the welfare of graph g, $W(g)$, equal the sum of the net utility of the agents:

$$W(g) = \sum_i u\left(s_i(g)\right) - c\sum_i \left(\sum_j g_{ij}\right).$$

That is, welfare is the difference between the total expected utility of agents in the network and the total link costs. We say a risk-sharing network g is *efficient* if it yields the highest welfare of all possible graphs. Formally, a network g is efficient if and only if there does not exist a network g' such that $W(g') > W(g)$.

We characterize efficient networks here and compare them to pairwise stable networks below. Let k denote the number of components of g, and let $s_1, ..., s_k$ denote the sizes of these components. Observe, first, components in efficient networks must be minimally connected. Removing any "redundant" link reduces costs by $2c$ without affecting benefits. We can then see that total number of links in a network g are a simple linear function of the number of components of the network and do not depend on their sizes or shapes. Let $|g|$ denote the number of links in network g. Since the number of links in a minimally connected component of size s is always equal to $s-1$, we have $|g| = \sum_{i=1}^{k}(s_i - 1) = n - k$. E.g., if there are n components, $k = n$, no individual has links, and $n-k = 0$. On the other extreme, if there is a single component, $k = 1$, the number of links is $n-1$, the minimal links necessary to connect the population in a single component. The total costs of links are then equal to $2c(n-k)$. Since every individual in the same component obtains the same risk-reducing benefits, aggregate benefits on a component of size s are equal to $su(s)$. We summarize these observations in the following lemma.

Lemma 1. *In an efficient network, g^*, components are minimally connected and welfare can be written as follows*

$$W(g^*) = \sum_{i=1}^{k} s_i u(s_i) - 2c(n-k)$$

We can then see that the shape of an efficient network - whether it should have none, few, or many components - depends on how component size affects the aggregate expected utility of agents in the component. Let $U = su(s)$ be this aggregated expected utility.

The functions $u(s)$ and $su(s)$ are defined over integers. To illustrate our results, we find it helpful to use extensions of these functions that are continuous and twice-differentiable. We also use these extensions for intermediate steps in proving our results. Below we will use derivatives, and the reader should understand that any derivative refers to a derivative of the continuous, twice differentiable extensions of our basic functions.

Let us consider the shape of networks that maximize aggregate expected utility $U = su(s)$. As component size s increases, there is a direct effect on aggregate utility, as an additional agent's utility is added to the sum, and there is an indirect effect as the increase changes other agents' utility. To illustrate, consider $u(s)$ that is continuous and twice-differentiable. We would then have

$$\frac{dU}{ds} = u(s) + su'(s)$$

Both of these effects are positive, as expected utility is always increasing in the size of the component ($u' > 0$). The question is then whether or not these positive effects are increasing enough to justify the cost of adding more agents to a component. That is, we must examine the derivative, $\frac{d^2U}{ds^2}$, how marginal benefits change as component size increases:

$$\frac{d^2U}{ds^2} = 2u'(s) + su''(s)$$

Since the expected utility function $u(s)$ is concave, $\frac{d^2U}{ds^2}$ could be positive or negative. When $-s\frac{u''(s)}{u'(s)} < 2$, the aggregate expected utility function U is convex, and when $-s\frac{u''(s)}{u'(s)} > 2$ the aggregate expected utility function U is concave.

Consider first convex aggregate expected utility. Marginal benefits are increasing in component size, and we have corner solutions. The efficient network will either involve one large component connecting all agents in the population or contain n components so that no agents have links. A network where all n agents are in a component yields net payoffs $n \cdot u(n) - 2(n-1)c$. If these payoffs are greater than the payoffs earned by individuals when they are all isolated, $n \cdot u(1)$, then the efficient network involves all n agents in a single component. That is, for $\frac{d^2U}{ds^2} > 0$, if $nu(n) - 2(n-1)c > nu(1)$, the efficient network will involve all n agents in the same component. Otherwise, no agents are connected in the efficient network.

Next consider concave aggregate expected utility. Marginal benefits are decreasing in component size, and the efficient network will involve smaller, intermediate size, components. An efficient network is composed, as much as possible, of components of size s such that $s^2u'(s) = 2c$.

The next result formally characterizes efficient risk-sharing networks. Let $c^* = \frac{n[u(n)-u(1)]}{2(n-1)}$ denote the cost level for which an empty network and a component connecting all n agents yield the same welfare.

Proposition 2. *If $su(s)$ is linear or strictly convex, an efficient network contains one component connecting all n agents if $c < c^*$ and contains no links if $c > c^*$.*

If $su(s)$ is strictly concave, the efficient network contains components of sizes that differ by at most one agent. For any $\varepsilon > 0$, if n is large enough, the average size component in an efficient network lies between $\hat{s} - \varepsilon$ and $\hat{s} + 1 + \varepsilon$ where $\hat{s}$ is the highest integer smaller than or equal to the solution of the equation $s^2u'(s) = 2c$.

This result tells us that, despite costly link formation and bilateral risk sharing, there are circumstances where the efficient network yields complete income pooling in the population. Hence any divergence from complete income pooling would come from a decentralized formation process. A quadratic primitive utility function yields this case and we provide an example below. When people are risk averse enough, the gains from connecting all individuals exceed the link costs. Otherwise, no-one should be connected.

Example 2. *When the primitive function is quadratic, $su(s) = sv(\overline{y}) - \lambda\sigma^2$, which is linear in s. Hence, the efficient network involves either everyone in a one component, or no risk sharing. The condition $c < c^*$ - all agents should be in a single component - simply reduces to $2c < \lambda\sigma^2$. Recall the parameter λ represents the level of risk aversion. When λ is high enough, all agents should be in a single component. Similarly, when the variance of incomes is high enough, all agents should be in a single component.*

When $su(s)$ is concave, there should be an intermediate level of risk-sharing: the costs of link formation impose a limit on the optimal size of connected components. Incomplete income pooling is a socially optimal outcome.

IV. A Static Link Formation Game

We now consider the formation of risk-sharing links. We ask what structures will emerge when pairs can agree to form links but agents cannot coordinate link formation across the whole population. That is, we have decentralized process of link formation. We then compare our outcomes to efficient networks.

We build a two-stage game. First, agents form links. To form a link, each partner must agree to the link and each must pay a cost c. This stage yields a network of links g. Second, income shocks are realized, and agents earn utility from the interaction process described above. The expected utility function u then gives expected payoffs for an agent in a network g. The net payoff to individual i of forming links equals

$$u(s_i(g)) - c\sum_j g_{ij}.$$

We solve for *pairwise stable* networks, an equilibrium concept developed by Jackson and Wolinsky (1996) for link formation games. The concept of pairwise stability specifies that agents must agree to form a link, but a single agent can break a link. Thus, a network is pairwise stable if and only if, there are no two agents who could gain by forming a new link, and there is no single agent who could gain by breaking a link. Let $g + ij$ denote the graph g with the addition of a link between agents i and j, and let $g - ij$ denote a graph g subtracting any link between agents i and j. Formally we have:

Definition 1. *A risk-sharing network g is pairwise stable iff*

(1)$\forall ij$ *s.t.* $g_{ij} = 0$, *if* $u(s_i(g + ij)) - c > u(s_i(g))$ *then* $u(s_j(g + ij)) - c < u(s_j(g))$

(2) $\forall ij$ *s.t.* $g_{ij} = 1$, $u(s_i(g)) - c \geq u(s_i(g - ij))$

The first condition says that, given others' links, no two agents want to form a new link. The second condition says that, given the set of links, no individual wants to unilaterally sever one of his links.

A. Pairwise Stable Networks

We now characterize pairwise stable networks. We find that the basic structure of pairwise stable networks is unique. For any population size n and link cost c, pairwise stable networks have a well-defined shape. First, there are no "extra" links in a pairwise stable network. If a link does not increase the size of a component, then it brings no benefits and an agent would have an incentive to cut the link. Second, pairwise stable networks divide agents into distinct components. Third, the size of the components is bounded. A component cannot be too large, because the benefits of linking to an individual in a larger and larger component eventually will not exceed the cost. A second component is smaller. If the second component is too large, then there is an incentive for a pair to make a link between components. But such a super-sized component is itself not stable. Hence, stable components will not generally include all members of the population. Finally, combining the second and third findings implies that pairwise stable networks do not always exist. In such a setting, we might expect to see cycles of networks.

We proceed by deriving successive restrictions on the shape of pairwise stable networks.

First, pairwise stable networks must have minimally connected components. This is due to the fact that benefits generated by a risk-sharing network only depend on the size of the components of the network. Therefore, individuals will sever any link that does not affect the size of the components.

Lemma 2. *In a pairwise stable network g, any component is minimally connected.*

Clearly, this property is shared by any network model where the benefits only depend on the size of the network components.

Second, components of a pairwise stable network cannot be too large. This result follows from the concavity of the expected utility function. As the size of a component increases, benefits increase but at a decreasing rate. Hence, there exists a threshold size s^* where the benefits of an additional agent do not exceed the cost:[13]

$$s^* = \max\{s : u(s) - u(s-1) \geq c\}$$

Lemma 3. *In a pairwise stable network g, all components have a size lower than or equal to s^*.*

This result follows from the second condition of pairwise stability - no agent can have an incentive to cut a link. In general to check that no individual wants to cut a link, it is necessary and sufficient to find the individual earning the least from a link and to check that he does not want to cut this link. In our setting, we can precisely define this individual. He is connected to a *peripheral* agent, which we define as an agent with a single link (a minimally connected

[13] Since u is concave, $u(s) - u(s-1)$ is decreasing and s^* is well-defined as soon as $u(2) - u(1) \geq c$. When $u(2) - u(1) < c$, only the empty network is pairwise stable.

graph always has at least two peripheral agents). Cutting a link to a peripheral agent reduces the component size by one, while cutting a link to a non-peripheral agent reduces the component size by more than one. Hence, the stability condition for cutting a link is derived from incentives to cut links to peripheral agents. In a pairwise stable graph, individuals must earn more with the link to a peripheral agent than without it.[14]

Third, we show that in pairwise stable networks, the size of the larger of two components must be exactly equal to s^*. This outcome arises from the first condition of pairwise stability - no two agents can want to form a link - together with the concavity of u.

Lemma 4. *For any two components of a pairwise stable network g, the size of the largest component is equal to s^*.*

The proof follows from the requirement that, in a pairwise stable network, no two individuals in the different components can want to form a link. Individuals in the largest component have a lowest incentive to do so, hence it is their payoffs that give us the stability condition. If the size of the largest component, s', is lower than s^*, individuals in the largest component would benefit from connecting to an individual in the smaller component. [At a component of size $s' < s^*$, individuals would gain from connecting to one individual, hence they would also gain from connecting to an individual connected to others.] Hence, along with Lemma 3, we have that the largest component must be equal to s^*.

To obtain the final characterization, we find the largest possible size of the smaller component. We define the threshold s^{**} as follows:

$$s^{**} = \max\{s \leq s^* : u(s^* + s) - u(s^*) < c\}$$

An agent in a component of size s^* would want to form a link to agent in a component larger than than s^{**}. Hence, any components larger than s^{**} cannot be part of a pairwise stable network alongside the component of size s^*. This threshold size s^{**} is well-defined and greater than or equal to 1 since by definition of s^*, $u(s^* + 1) - u(s^*) < c$.

We then have our result which completely characterizes pairwise stable networks. For any

[14] More generally, individual i earns less from a link with j if the number of indirect neighbors of j in the graph without the link ij is lower.

population size n and link cost c, pairwise stable networks have a unique shape:

Proposition 3. *When $s^{**} < s^*$ (unequal components case) a risk-sharing network g is pairwise stable if and only if: (1) if $n \leq s^*$, it is minimally connected; and (2) if $n > s^*$, it is composed of two minimally connected components of sizes s^* and $s \leq s^{**}$ When $s^{**} = s^*$(equal components case) a risk-sharing network is pairwise stable iff (1) all its components are minimally connected; (2) all its components except* 1 *have size s^*; and (3) the size of the remaining component is lower than or equal to s^*.*

While the basic structure of pairwise stable networks is unique, they might not always exist. Existence depends on the relationship between the necessary component sizes and the population size. The equal components case appears if the utility function u is sufficiently flat above s^*. In this case, we can have $u(s^*) - u(s^* - 1) \geq c$ and $u(2s^*) - u(s^*) < c$, and pairwise stable networks exist for all values of n. In other cases, the existence of pairwise stable networks may be *non-monotonic* in the link formation cost c. A decrease in c has two effects. First, it may lead to a increase in s^*; the largest component is higher since it is less costly to maintain a link to a peripheral agent. Second, it can decrease s^{**}, since building links between components is less costly. Thus, for a certain population level n, there could exist three values $c_1 > c_2 > c_3$, such that a pairwise stable network exists for c_1 and c_3, but not for c_2. For the high cost level, there is a small difference between the size of the components, and for the low cost level there is a higher difference. Thus higher link costs can equalize risk-sharing outcomes. In section V, we discuss what happens when pairwise stable networks do not exist.

We illustrate Proposition 3 with our quadratic example.

Example 3. *Consider, again, a quadratic primitive utility function which yields the expected utility function $u(s) = v(\overline{y}) - \frac{\lambda\sigma^2}{s}$, and set $\lambda\sigma^2 = 1$. The table below gives the threshold values s^* and s^{**} as functions of the cost of link formation c. The third row gives the maximum population size, $\overline{n}$, for which there exists a pairwise stable graph. We have $\overline{n} = \infty$ when $s^* = s^{**}$, and $\overline{n} = s^* + s^{**}$ when $s^* < s^{**}$. When $c > \frac{1}{2}$, the link cost exceeds the benefit of any risk-sharing,* *hence $s^* = s^{**} = 1$ and the empty network is always stable.*

Values of c	$c > \frac{1}{2}$	$\frac{1}{2} \geq c \geq \frac{1}{4}$	$\frac{1}{4} > c > \frac{1}{6}$	$c = \frac{1}{6}$	$\frac{1}{6} > c \geq \frac{2}{15}$	$\frac{2}{15} > c > \frac{1}{12}$
(s^*, s^{**})	$(1,1)$	$(2,2)$	$(2,1)$	$(3,3)$	$(3,2)$	$(3,1)$
Max n with pws graph	∞	∞	3	∞	5	4

Values of c	$c = \frac{1}{12}$	$\frac{1}{12} > c > \frac{1}{20}$	$\frac{1}{20} \geq c > \frac{1}{30}$	$\frac{1}{30} \geq c > \frac{1}{42}$
(s^*, s^{**})	$(4,2)$	$(4,1)$	$(5,1)$	$(6,1)$
Max n with pws graph	6	5	6	7

We see here that as c decreases, the critical size of the largest component increases. And as c increases, there is greater gap between the size of the smallest and largest component.

B. Pairwise Stable vs. Efficient Network

Finally, we compare pairwise stable and efficient networks. In general, there are two sources of divergence between pairwise stable and efficient networks. First, there could be a pairwise inefficiency: a link between agents i and j could increase their joint payoffs by $2c$, but the benefits are not equal so i is willing to pay c, but j is not. In our model, a link between i and j increases the sizes of their components. For i the increase is from s_i to $s_i + s_j$; for j the increase is from s_j to $s_i + s_j$. Since i and j can start in different size components, they do not have the same benefit from the link. It is then possible that the link is worth more than $2c$ to both agents, but less than c to agent j.

$$u(s_i + s_j) - u(s_j) - c < 0 < 2u(s_i + s_j) - u(s_i) - u(s_j) - 2c$$

This case can arise when agent i is a peripheral agent. From a social welfare point of view the link should be formed, but j does not have the incentive to do so. Second, there could be a global inefficiency: a link between agent i and agent j can benefit others, but i and j do not benefit enough to both pay the link cost. In our model, the increase in utility for i and j from being in larger components does not exceed the link costs, but the link increases overall welfare:

$$2u(s_i + s_j) - u(s_i) - u(s_j) - 2c < 0 < W(g + ij) - W(g)$$

This case can arise when a link between components would increase overall welfare, but not the joint payoff for the pair. As both types of inefficiency arise in our model, pairwise stable networks in general involve (weakly) smaller components than efficient networks. We can show this result directly, by comparing Propositions 3 and 2:

Proposition 4. *If $su(s)$ is strictly convex, or if $su(s)$ is strictly concave and n is high enough, the size of components in efficient networks is always greater than the size of components in pairwise stable networks.*

This inefficiency is severe in the case where the efficient network involves a single component connecting all the agents. Recall that when aggregate expected utility is convex, the efficient network connects all individuals when $n \cdot u(n) - 2(n-1)c \geq n \cdot u(1)$, which can be written $\frac{u(n)-u(1)}{n-1} \geq \frac{2c}{n}$. That is, a social planner would want to increase the component size to n as long as the average increase in benefits exceeds the average cost. In this case, the efficient network would yield risk-sharing that mimics complete income pooling within the population. However, the pairwise stable network will include components only of size s^* and of smaller sizes, where, recall, $s^* = \max\{s : u(s) - u(s-1) \geq c\}$. That is, an individual would maintain a link only when the marginal benefits exceed the cost. Hence, due to the decentralized link formation process, risk-sharing networks would yield outcomes that look like incomplete income pooling within a population, as well as inequalities in risk-sharing outcomes.

When the aggregate expected utility function is concave, the efficient network involves intermediate size components. A social planner again would choose a network considering the change in average benefits and average costs. But since, individuals only consider their own benefits, the components in a pairwise stable network are smaller than in efficient networks. And in pairwise stable networks, individuals can be in different size networks. Individuals do not have symmetric outcomes.

We illustrate with an example using a quadratic utility function:

Example 4. *Consider, again, a quadratic primitive utility function which yields the expected utility function $u(s) = v(\overline{y}) - \frac{\lambda\sigma^2}{s}$ and set $\lambda\sigma^2 = 1$. Example 3 solves for the pairwise stable networks for different cost levels. Let us consider $c = \frac{1}{4}$. As calculated above, for any size population, a pairwise stable network exists and consists of components of size 2 (subject to integer* *constraints). The efficient network, in contrast, places all individuals in a single component. Example 2 showed that when $2c < \lambda\sigma^2$, this network form is efficient. Here we have $2(\frac{1}{4}) < 1$, which satisfies this condition. Thus, the decentralized network formation leads to very small connected components and limited income pooling relative to the efficient outcome.*

V. A Dynamic Link Formation Model

In this section, we consider a dynamic model of network formation. We ask what networks emerge when agents can continually make and break links. We do so in order to gain insights into the evolution of risk-sharing networks.

Starting from an initial set of risk-sharing relations, what type of network emerges? Consider the following stochastic process from Jackson and Watts (2002).[15] Start with an initial network g^0. At each time t, a pair (i, j) is picked with probability p_{ij} where $\forall(i, j), p_{ij} > 0$ and $\sum_{(i,j)} p_{ij} = 1$. When the pair is picked, they decide either to make a link if they are not already connected, or any one of the pair can break the link if they are connected. That is, if $g^t_{ij} = 1$, each considers whether she wants to keep the link. If $u(s_i(g^t - ij)) > u(s_i(g^t)) - c$, i would be better off without the link and the link is cut. Similarly for j. The network evolves, and we have $g^{t+1} = g^t - ij$ when the link is cut. Otherwise, the network remains unchanged. If $g^t_{ij} = 0$, the agents both consider whether they want to form the link. For instance, if $u(s_i(g^t + ij)) - c > u(s_i(g^t))$ and $u(s_j(g^t + ij)) - c \geq u(s_j(g^t))$, the link is formed. Similarly if $u(s_j(g^t + ij)) - c > u(s_j(g^t))$ and $u(s_i(g^t + ij)) - c \geq u(s_i(g^t))$. In this case, the network evolves and $g^{t+1} = g^t + ij$.

This defines a stochastic, dynamic process of network formation. Links are formed or cut one at a time, in a myopic fashion. We ask if there is any prediction as to the outcome of this process.

We use Jackson and Watts' (2002) notions of improving paths and closed cycles to characterize the outcomes. There is an *improving path* from g to g' if and only if there is a strictly positive probability that starting at g, the dynamic process leads to g'. A *closed cycle* C is a set of networks such that for any two networks $g, g' \in C$, there is an improving path from g to g' and improving paths from a network in C can only lead to another network in C. If g is pairwise stable, $\{g\}$ is a closed cycle.

[15] Here, income shocks are realized and agents share risk *between* any two periods where the network evolves. The many rounds of income-sharing (as considered in section II) take place when the links are fixed; that is, between the periods when agents can change links.

The results confirm the insights from the static game. The dynamic process yields pairwise stable networks, when pairwise stable networks exist. When pairwise stable networks do not exist, we see cycles among different networks. The cycles, though, include only certain kinds of networks. In particular, the networks are minimally connected and always include networks with small components, of size less than or equal to s^*. Hence, we see that the networks that emerge from this process are often not socially efficient. They connect too few people. And agents on the periphery of networks are vulnerable - links to them are cut more often.

Our formal results follow:

First, if there is a pairwise stable network, the dynamic process eventually yields a pairwise stable network. No other outcome is possible.

Proposition 5. *If a pairwise stable network exists, there is no other closed cycles.*

Corollary 1. *If a pairwise stable network exists, starting from any network, the dynamic process converges to a pairwise stable network with probability* 1.

Second, when no pairwise stable network exists, the networks that emerge from this dynamic process are well-defined. They are all minimally connected and may have small size components, less than or equal to s^*.

Proposition 6. *If a pairwise stable network does not exist, there is a unique closed cycle C. Any network in C has minimally connected components, and has one component of size greater than or equal to s^*. In addition, all networks with minimally connected components of sizes $(s^*, ..., s^*, s')$ where $s' \leq s^*$ belong to C.*

In this case, there are cycles, where individuals make and break links forming new networks.

Corollary 2. *If a pairwise stable network does not exist, starting from any network, the dynamic link formation process leads to a network in C with probability* 1. *As soon as a network in C is reached, all subsequent networks are also in C.*

Third, we see a further illustration of the vulnerability of peripheral agents. In this dynamic process, individuals with fewer links are more likely to be cut out of a component. They bring less benefits than other agents, and links to them are less valuable.

Proposition 7. *Within C, as soon as a link is deleted, it happens in a component of size strictly greater than s^* and to an individual that has less than the average number of indirect neighbors.*

To summarize, suppose that we begin in a situation where all individuals are isolated. At first, individuals connect with each other and the size of the networks' components grows. As soon as the size is greater than s^*, the dynamics results from two countervailing forces. First, neighbors of individuals situated at the periphery would want to sever their links to them. Second, individuals want to form links to reap risk-sharing benefits. The size of connected components will then grow and shrink over time.

VI. Conclusion

This paper develops the first model of network formation where links are used to share risk. We model a benchmark case where individuals share income equally with their linked partners and, hence, complete insurance can be attained in a component of the network. That is, individuals earn the average income of agents in the component, and an agent would prefer to be in larger components since there is lower variability of the mean income

We ask what structures will emerge when pairs can agree to form links but agents cannot coordinate link formation across the whole population. We compare efficient networks - which maximize aggregate expected utility - and pairwise stable networks - which are formed by individuals acting to maximize their own expected utility.

We find, first, that efficient networks can (indirectly) connect all individuals and involve full insurance, despite the cost of links. Pairwise stable networks, however, connect fewer individuals. There is an externality: when breaking a link individuals do not take into account the negative effect on others distant in the network. Peripheral agents - agents with no links to others - are especially vulnerable to being cut out of a component.

Second, the network formation process can lead ex ante identical individuals to be different positions ex post and thus have different risk-sharing outcomes. These results may help explain empirical findings that risk-sharing within a similar population is often not symmetric or complete.

In our static model, stable networks may not exist. If components are too large, links to peripheral agents are cut. But if components are too small, or agents are isolated, they have an incentive to form links to others. What happens in this case? In our study of the dynamic model, we show that cycles may emerge. Especially, during the dynamic making and breaking of links, peripheral agents are vulnerable. They are likely to be cut out of a connected component. This occurs because network components cannot stay large for a long time. In large components, agents have an incentive to cut links when they have the chance to do so. Hence, even in the long term, we do not expect that agents, acting on their own to establish risk-sharing relations, will form efficient risk-sharing networks.

Frictions in the income sharing process is a direction for future research. Our current model has no frictions: we show that if individuals commit to share monetary holdings equally within pairs, and pairs interact repeatedly, in the limit individuals' incomes are shared equally within components. In this process, individuals do not need to have information on the past transfers between people. They only need to know the current level of monetary holdings of their neighbors when they meet them. There is also no discounting or other income losses that occur as these interactions proceed.

This benchmark model then indicates several ways to introduce frictions into the risk-sharing process. If the process takes place over a finite number of rounds, monetary holdings within the component will, in general, not equalize within the component, but depend on an agent's position within the component. A more connected person might meet more often with neighbors and have greater risk sharing. If individuals can only partially observe net monetary holdings, then only some parts of income and transfers can be shared. Finally, if some income is consumed before all transfers are complete, expected utility will not be equalized within a component. With these frictions, agents' utility would depend not only on the size of their components, but their positions in the component. We expect that pairwise stable networks are no longer minimally connected, as people gain more from direct connections than from indirect connections. Yet, positive externalities will remain.

Thus, we expect the central effects that we identify in this paper will hold. Peripheral individuals will be vulnerable because links that connect them to others are less valuable than links to better connected agents. Pairwise stable networks will connect fewer individuals than efficient networks. And the decentralized link formation process can lead to asymmetric outcomes for otherwise similar individuals.

Appendix

Proof of Proposition 1 Without loss of generality, suppose that the network is connected. (The result then applies to any component of a network). If $y = (y_1, ..., y_n)$ is a vector of income, define $d(y) = \sum_i (y_i - \bar{y})^2$ as the dispersion of y around the mean $\bar{y} = \frac{1}{n}\sum_{i=1}^n y_i$. The following property will be useful.

Lemma: Consider two neighbors i and j with $y_i \neq y_j$. When i and j share income equally, $d(y)$ decreases strictly.

Proof: If $\tilde{y}$ denotes the vector of incomes after transfers, we have $\tilde{y}_i = \tilde{y}_j = \frac{1}{2}(y_i + y_j)$. This yields

$$d(\tilde{y}) - d(y) = 2[\frac{1}{2}(y_i + y_j) - \bar{y}]^2 - (y_i - \bar{y})^2 - (y_j - \bar{y})^2 = -\frac{1}{2}(y_i - y_j)^2$$

This shows that d is strictly lower as soon as $y_i \neq y_j$.□

Denote by y^T the vector of incomes at round T and by y^0 the initial vector of incomes. The vector y^{T+1} is obtained from y^T by a certain function Φ_T, which is picked at random at every round within a set of functions. This set is finite since pairs do not interact more than M times at each round. A function in this set depends on the order at which pairs of neighbors interact. Thus, $y^{T+1} = \Phi_T(y_T)$. Since Φ_T consists in a series of bilateral equalizations, by the previous lemma, $d(y^{T+1}) \leq d(y^T)$. In fact, since all pairs of neighbors interact at least once, $d(y^{T+1}) < d(y^T)$ as soon as $y^{T+1} \neq y^T$. We will now show that $d(y^T)$ converges to 0, which implies that y_i^T converges to $\bar{y}$.

We next make use of the following classic topological property. In a compact set, any sequence admits a subsequence which converges to an element of this compact set. Let $y_{\max}$ be the largest possible income realization and define $Y = [0, y_{\max}]^n$. Since $y^T \in Y$ and Y is compact, we know that y^T has a subsequence $y^{f(T)}$ which converges to $y^* \in Y$. Since d is continuous, $d(y^{f(T)})$ converges to $d(y^*)$. Since $d(y^T)$ is non-increasing, we know that $d(y^T)$ converges to $d(y^*)$.

Consider now the sequence $y^{f(T)+1}$. Applying the same topological property, we know that it has a subsequence $y^{f(g(T))+1}$ which converges to $y^{**} \in Y$. Moreover, since $d(y^T)$ converges to $d(y^*)$, $d(y^{**}) = d(y^*)$. By definition, $y^{f(g(T))+1} = \Phi^{f(g(T))}(y^{f(g(T))})$. Since the number of possible functions Φ is finite while the sequence $f(g(T))$ is infinite, there exists one function represented an infinite number of times in the sequence $\Phi^{f(g(T))}$. Thus, there exists a fixed function Φ and a subsequence h such that $y^{f(g(h(T)))+1} = \Phi(y^{f(g(h(T)))})$. We can now take the limit on both sides.

It yields $y^{**} = \Phi(y^*)$. Since $d(y^{**}) = d(y^*)$, it means that $y^* = y^{**}$, which is only possible when $\forall i, y_i^* = y_i^{**} = \bar{y}$. Thus, $d(y^*) = 0$, hence $d(y^T)$ converges to 0.

Proof of Proposition 2 Suppose first that $su(s)$ is strictly convex: $2u' + su'' > 0$. Welfare is maximized when all components except one are singletons. It becomes

$$W(k) = (k-1)u(1) + (n-k+1)u(n-k+1) - 2c(n-k)$$

Deriving once with respect to k yields $W' = u(1) + 2c - u(n-k+1) - (n-k+1)u'(n-k+1)$. Deriving again yields $W'' = 2u'(n-k+1) + (n-k+1)u''(n-k+1)$ which is positive. Thus, W is strictly convex and the maximum is at a corner, i.e., either $k = 1$ and the network is completely connected, or $k = n$ and the network is empty.

Suppose next that $su(s)$ is linear, so that $u(s) = u_\infty - \lambda/s$. Welfare becomes

$$W(g) = nu_\infty - 2cn + (2c - \lambda)k$$

which is a linear function of k. It is then enough to check that $c^* = \lambda/2$.

Finally, suppose that $su(s)$ is strictly concave, $2u' + su'' < 0$. Components will have, as much as possible, equal size. More precisely, if $n = sk + t$ with s, t integers and $0 \leq t < k$, then $k - t$ components have size s and t components have size $s + 1$. Define the average welfare $w(g) = \frac{1}{n}W(g)$ and $w(k)$ the highest average welfare over graphs with k components. Introduce the function $f(x) = u(\frac{1}{x}) - 2c(1-x)$ for any $x \in]0, 1]$. Solving the welfare maximization problem with $s_i \in R$ yields $s_i = \frac{n}{k}$, hence $w(k) \leq f(\frac{k}{n})$ with equality only when k divides n. The derivatives of f are $f'(x) = -\frac{1}{x^2}u'(\frac{1}{x}) + 2c$ and $f''(x) = \frac{1}{x^3}[2u'(\frac{1}{x}) + \frac{1}{x}u''(\frac{1}{x})] < 0$. Since f is concave, it attains its maximum in $\frac{1}{\tilde{s}}$ where $\tilde{s}$ solves $s^2u'(s) = 2c$.

Next, for any given s, construct a graph g as follows. All component except 1 have size s and the remaining component has size lower than or equal to s. Write $n = \alpha s + \beta$ where α, β integers and $0 < \beta \leq s$. The graph g has $\alpha + 1$ components. Average welfare of g is $w(g) = \frac{1}{n}[\alpha su(s) + \beta u(\beta)] - 2c(1 - \frac{\alpha+1}{n})$ which can be rewritten as:

$$w(g) = \left(1 - \frac{\beta}{n}\right)u(s) + \frac{\beta u(\beta)}{n} - 2c(1 - \frac{1}{s}) + 2c\frac{s-\beta}{ns}$$

This shows that $w(g) \to f(\frac{1}{s})$ as $n \to \infty$.

Let $\hat{s}$ be the highest integer lower than or equal to $\tilde{s}$. Since the maximum of f lies between $\frac{1}{\hat{s}+1}$ and $\frac{1}{\hat{s}}$, f is concave, and $w(k) \leq f(\frac{k}{n})$, we know that there for any $\varepsilon > 0$ there exists n_0 such that $n > n_0$ implies that the k maximizing $w(k)$ is such that $\frac{k}{n}$ belongs to $[\frac{1}{\hat{s}+1} - \varepsilon, \frac{1}{\hat{s}} + \varepsilon]$. Equivalently, for any $\varepsilon > 0$ there exists n_0 such that $n > n_0$ implies that the k maximizing $w(k)$ is such that $\frac{n}{k}$ belongs to $[\hat{s} - \varepsilon, \hat{s} + 1 + \varepsilon]$.

Proof of Proposition 3

Proof of Lemma 3: Consider a minimally connected component of size s of a pairwise stable network g. For a link ij in the component, condition (1) is satisfied iff $u(s_i(g - ij)) \leq u(s) - c$. Thus, condition (1) is satisfied for all the links ij iff it is satisfied for the links ij with highest $s_i(g - ij)$. It is easy to see that $s_i(g - ij) \leq s - 1$ with equality if and only if j is a peripheral agent. Thus, condition (1) is satisfied for all the links if and only if $u(s) - u(s - 1) \geq c$. □

Proof of Lemma 4: Consider a pairwise stable network g with two components of sizes s and s' such that $s \leq s'$. Since $u(s) \leq u(s')$, condition (2) is satisfied for every pair of agents in the two components iff $u(s+s')-u(s) < c$. Since $u(s+s')-u(s+s'-1) \leq u(s+s')-u(s)$, we have $s+s' > s^*$. Next suppose that $s' < s^*$. We have $u(s+s') - u(s) \geq u(s^*) - u(s) \geq u(s^*) - u(s^* - 1) \geq c$ which is a contradiction. Therefore $s' = s^*$. □

Finally, suppose that there are k components with $k \geq 3$. Lemma 4 implies that the $k - 1$ larger components have a size equal to s^*. In this case, the network is pairwise stable only if $u(2s^*) - u(s^*) < c$, which means that $s^{**} = s^*$.

Proof of Proposition 4

When U is linear or convex, the result follows directly from Proposition 3. Suppose, then, that U is concave. If $s^* = 1$, the empty network is the only stable network, so we can assume that $s^* \geq 2$. Choose an extension of u such that $u'(s^*) = c$. Let $\tilde{s}$ be such that $s^2 u'(s) = 2c$ and $\hat{s}$ be the highest integer lower than or equal to $\tilde{s}$. Then,

$$s^{*2} u'(s^*) > s^{*2} c \geq 4c > 2c = \tilde{s}^2 u'(\tilde{s})$$

Introducing $g(x) = x^2 u'(x)$, we see that $g'(x) = x(2u'(x) + xu''(x)) < 0$ by concavity of U. Hence g is strictly decreasing and $s^* < \tilde{s}$, hence $s^* \leq \hat{s}$. By Proposition 2, we know that if n is large enough, the two possible component sizes in efficient networks are $(\hat{s} - 1, \hat{s})$, $(\hat{s}, \hat{s} + 1)$, or

$(\hat{s}+1, \hat{s}+2)$. Thus, the size of the larger components in efficient networks is always greater than or equal to s^*.

Proof of Propositions 5, 6, 7.

Observe that redundant links can be cut, but can never be formed. This implies that any network in a closed cycle has minimally connected components and we now restrict attention to these networks.

Next, from any network g, we construct an improving path to a network whose components have size $(s^*, ..., s^*, s')$ where $s' \leq s^*$. For any two components with sizes s_i and s_j such that $s_i, s_j < s^*$, we have $u(s_i + s_j) - u(s_i) \geq u(s_i + 1) - u(s_i) \geq c$, hence there is an improving path to a unique component of size $s_i + s_j$. Repeating the operation leads to a network with at most one component of size s strictly less than s^* and all the others with sizes greater than or equal to s^*. Then, from any component of size strictly greater than s^*, a link with a peripheral agent can be cut. This can be repeated til the size of the larger remaining component is s^* and all other individuals are isolated. Doing this on any large component leads to a network with one component of size s and others of sizes s^* and 1. Links can then be formed between isolated individuals and the smaller component and between isolated individuals to lead to a network with components of sizes $(s^*, ..., s^*, s')$ where $s' \leq s^*$. This shows that any closed cycle must possess one such network.

Next, suppose that a pairwise stable network exists. There are two cases. First, $s^{**} < s^*$ and $n \leq s^{**} + s^*$. By the previous argument, from any network, there is an improving path to a network with minimally connected components of sizes s^* and $s' \leq s^*$. Since $n = s^* + s'$, $s' \leq s^{**}$, and this network is pairwise stable. Second, $s^{**} = s^*$, and the network constructed in the previous reasoning is also pairwise stable. In either case, the pairwise stable networks are the only closed cycles.

Suppose that a pairwise stable does not exist. It means that $s^{**} < s^*$ and $n > s^{**} + s^*$. From any two networks g and g' whose components have size $(s^*, ..., s^*, s')$ and $s' \leq s^*$, we construct an improving path from g to g'. Observe that from s^* isolated individuals, there is an improving path to *any* minimally connected network of size s^*. Also, since $u(2s^*) - u(s^*) \geq c$, from two components of size s^*, there is an improving path to a component of size $2s^*$, hence to a network with one component of size s^* and s^* isolated individuals, hence to a network with two components of size s^* where one of the two has arbitrary structure. Similarly, since $s' > s^{**}$, from a networks with two components of size s^* and s' there is an improving path to a network with two components of the same sizes, where the component of size s' has arbitrary structure. Repeating this operation enough times, there is also an improving path to a network with two components of the same size, where the component of size s^* has arbitrary structure. Combining these properties, from any two networks g and g' whose components have size $(s^*, ..., s^*, s')$ and $s' \leq s^*$, there is an improving path from g to g'. Since any closed cycle must possess a network with components of sizes $(s^*, ..., s^*, s')$, the previous property implies that there is a unique closed cycle.

A link is deleted in a component of size s when there exist i and j in the component such that $u(s) - u(s_i(g - ij)) < c$ where $s_i(g - ij) \geq s_j(g - ij)$. This implies that $s > s^*$ and also that $s_i(g - ij) \geq s^*$. Individual i wants to cut her link with j because this link does not bring enough payoff. In other words, $s_j(g - ij)$ is too low. This number $s_j(g - ij)$ measures how indirectly well-connected is j without his link to i. It equals 1 when j is a peripheral individual.

It is sufficient to show that, starting from a network with components of sizes $(s^*, ..., s^*, s')$, improving paths can only lead to networks with at least one component of size greater than or equal to s^*. Addition of links preserve this property. We just showed that deletion of links can only take place in components of size strictly greater than s^* and that after the deletion, a component of size greater than or equal to s^* always remains. Thus, the property is also preserved by link deletions. This shows that in any network in the closed cycle, one component has size greater than or equal to s^*.

References

Bloch, Francis, Genicot, Garance, and Ray, Debraj (2004): "Informal Insurance in Social Networks," working paper.

Coate, Stephen and Ravallion, Martin. (1993): "Reciprocity without Committment: Characterization and Performance of Informal Insurance Arrangements," *Journal of Development Economics*, 40, 1-24.

Dercon, Stefan and De Weerdt, Joachim. (2005): "Risk-Sharing Networks and Insurance against Illness", forthcoming *Journal of Development Economics.*

De Weerdt, Joachim. (2004): "Risk-Sharing and Endogenous Network Formation", chapter 10 of *Insurance Against Poverty*, ed. by S. Dercon, Oxford: Oxford University Press.

Fafchamps, Marcel and Lund, Susan (2003): "Risk Sharing Networks in Rural Philippines", *Journal of Development Economics,* 71, 261-87.

Genicot, Garance and Ray, Debraj (2003): "Endogenous Group Formation in Risk Sharing Arrangements," *Review of Economic Studies*, 70 (1), January 2003, 87-113.

Jackson, M., and A. Watts (2002): "The Evolution of Social and Economic Networks," *Journal of Economic Theory*, 106, 265-295.

Jackson, M., and A. Wolinsky (1996): "A Strategic Model of Social and Economic Networks," *Journal of Economic Theory*, 71, 44-74.

Kimball, Miles (1988): "Farmer Cooperatives as Behavior Towards Risk," *American Economic Review*, 78, 224-232.

Kirman, Alan and Zimmermann, J.B. (eds.) (2001). *Economics with Heterogeneous Interacting Agents*. Berlin, Springer Verlag.

Kirman, Alan (1993). "Ants, rationality and recruitment," *Quarterly Journal of Economics*, 108 (1), February 1993, 137-56.

Ligon, E., Thomas, J., and Worrall, T. (2002): "Mutual Insurance and Limited Committment: Theory and Evidence in Village Economies," *Review of Economic Studies*, 69, 115-139.

Rosenzweig, Mark R. and Stark, Oded. (1989) "Consumption Smoothing, Migration, and Marriage: Evidence from Rural India," The Journal of Political Economy, 97(4), August 1989, 905-926.

Rothschild, M., and J. Stiglitz (1971): "Increasing Risk II: Its Economic Consequences," *Journal of Economic Theory*, 3, 66-84.

Townsend, Robert (1994): "Risk and Insurance in Village India", *Econometrica*, 62(3), May 1994, 539-591.

Udry, Christopher (1996): "Risk and Saving in Northern Nigeria," *American Economic Re-*

view.