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Bi-Polarization Comparisons

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Abstract: This note provides simple tests for first-order bi-polarization orderings of distributions of living standards. In doing so, the paper also offers an ethical basis and an interpretation for the common use of some simple measures of distances from the median. Illustrations using Luxembourg Income Study data show that several countries can be ranked by such tests of bi-polarization

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1 Introduction

The last decade has seen a surge of interest in the theoretical and empirical measurement of polarization – loosely speaking, the clustering of incomes around local poles.[1] The recent literature has also provided tools for ordering distributions over classes of bi-polarization indices, that is, over classes of indices that exhibit an ethical ”preference for the middle”[2]. This note builds on that literature by deriving simple tests for first-order bi-polarization orderings of distributions of living standards (Section 2). This differs from the earlier and recent work that concentrated on second-order bi-polarization orderings. The paper (Section 3) also offers an ethical basis for the common use of simple measures of distances from the median, thus allowing a re-interpretation of some of the simple indicators that could otherwise be described by (see for instance, Wolfson (1994), p.354) as ”unsatisfactory” and ”incoherent”.

Focussing on first-order (as opposed to second-order) bi-polarization orderings has the advantage of greater generality regarding the ethical properties of the bi-polarization indices, thus enabling searches for ”more unanimous” orderings. Such a focus can, however, limit the ordering power of the resulting tests. Illustrations (Section 4) using cross-country Luxembourg Income Study data nevertheless show that the first-order tests proposed by this paper seem empirically quite powerful in ordering many countries in terms of bi-polarization.

The proofs are found in the Appendix along with a sketch of the sampling distribution of the statistics used in the dominance tests.

2 Polarization Dominance

2.1 Measuring bi-polarization

Let $x = (x_1, x_2, \ldots, x_n) \in \Re^n_{++}$ be an n-dimensional vector of positive incomes, ordered in increasing values such that $x_1 \leq x_2 \leq \ldots \leq x_n$, and letting x_i be the income of the ith person. We assume that n is even – we will see later using the population-replication axiom 2 that this assumption is without consequence here. Median income is thus $m_x = x_{n/2}$. Let $d_x = (d_x(1), d_x(2), \ldots, d_x(n)) \in$

[1]See, among many others, Davis and Huston (1992), Foster and Wolfson (1992), Esteban and Ray (1994), Wolfson (1994), Jenkins (1995), Esteban and Ray (1999), Gradin (2000), Chakravarty and Majumder (2001), and Zhang and Kanbur (2001).

[2]See in particular Foster and Wolfson (1992), Wolfson (1994) and Wang and Tsui (2000).

$\Re^n_{++}$, with $d_x(i) = |1 - x_i/m_x|$. $d_x(i)$ is thus the proportional ”spread” of i’s income from the median. Then:

Definition 1 *A bi-polarization index $P(d_x) : \Re^n_{++} \rightarrow \Re$ is a function of the differences $d_x(i)$, $i = 1, \ldots, n$ of x_i from the median income m_x.*

Axiom 1 (Homogeneity) *The index P is homogeneous of degree zero in d_x, viz, for any $\gamma > 0$, we have*

$$P(d_x) = P(\gamma d_x). \tag{1}$$

Axiom 2 (Population invariance) *Adding a replication of a distribution x to that same distribution has no impact on P.*

We can therefore suppose, for the sake of expositional simplicity, that all income vectors x are of the same dimension n.

Axiom 3 (Monotonicity) *For a given n and a given m_x, the index $P(d_x)$ is monotonically increasing in the distance d_x. In other words, for any x and y in $\Re^n_{++}$ such that $m_x = m_y$, with $d_x(i) \geq d_y(i)$ for all $i = 1, \ldots, n$ and with $d_x(i) > d_y(i)$ for some $i = 1, \ldots, n$, then $P(d_x) \geq P(d_y)$.*

A simple index which obeys all of the above axioms is given by

$$Q_x(\lambda) = n^{-1} \sum_{i=1}^{n} I(d_x(i) \geq \lambda) \tag{2}$$

where $I(\cdot)$ is an indicator function that takes the value 1 if its argument is true and 0 otherwise. $Q_x(\lambda)$ shows the proportion of the population whose proportional distance from the median exceeds λ – loosely speaking, an index of ”bipolarity”.[3]

2.2 First-order bi-polarization dominance

Now define the class $\mathcal{C}(\mathrm{A1}, \mathrm{A2}, \mathrm{A3})$ as the class of all polarization indices $P(\cdot)$ which obey Axioms 1, 2 and 3. Then:

Theorem 1 (First-order bi-polarization dominance)

$$P(d_x) \geq P(d_y), \ \forall P(\cdot) \in \mathcal{C}(\mathrm{A1}, \mathrm{A2}, \mathrm{A3}) \tag{3}$$

$$\textit{iff } d_x(i) \geq d_y(i), \ \forall i = 1, ..., n. \tag{4}$$

[3] See, for instance, Morris, Bernhardt, and Handcock (1994).

Proof. See appendix. ■

Let d_x^* be the vector d_x rearranged in *increasing* values of the $d_x(i)$. A condition that is regularly implicitly imposed on bi-polarization indices derives from the following symmetry axiom:

Axiom 4 (Symmetry) $P(d_x) = P(d_x^*)$ *for any* $x \in \Re_{++}^n$.

Axiom 4 says that ignoring whether distances from the median occur on the right or on the left of the median should not matter for the measurement of bi-polarization. We then have:

Theorem 2 (First-order symmetric polarization dominance)

$$P(d_x) \geq P(d_y), \; \forall P(\cdot) \in \mathcal{C}(\text{A1}, \text{A2}, \text{A3}, \text{A4}) \tag{5}$$
$$\textit{iff } d_x^*(i) \geq d_y^*(i), \; \forall i = 1, ..., n. \tag{6}$$

Proof. The proof follows from the same arguments as for Theorem 1, this time using d_x^* and d_y^* as opposed to d_x and d_y. ■

It can be seen by inspection that (6) is a weaker condition than (4). It is easier to order bi-polarization indices that consider distances from the median symmetrically than over those that do not.

3 Discussion

Consider the following two income distributions expressed as a proportion of the median m_x:

$$x = (2m_x/3, 2m_x/3, 2m_x/3, m_x, m_x, m_x, m_x, 2m_x, 2m_x, 2m_x)$$
$$y = (0, 0, 0, 0, m_x, m_x, 2.5m_x, 2.5m_x, 2.5m_x, 2.5m_x).$$

Moving from distribution x to distribution y illustrates an increased ”spreads” effect: both the poor and the rich are getting more distant from the median. Figure 1 displays this. Individual i appears on the horizontal axis at percentile $p = i/n$ and the proportional distance from the median is shown on the vertical axis. The relative differences from the median are smaller in x than in y whatever the percentiles considered.

Inverting the distance curves provides an equivalent test that is interpretable in terms of a ”bi-polarity” criterion. For this, let $d_{x^-} = (d_x(1), d_x(2), \ldots, d_x(n/2))$

and $d_{x^+} = (d_x(n/2+1), d_x(n/2+2), \ldots, d_x(n))$. It can be checked that condition (4) is then equivalent to

$$Q_{x^-}(\lambda) \geq Q_{y^-}(\lambda),\ \forall \lambda > 0 \tag{7}$$

$$\text{and } Q_{x^+}(\lambda) \geq Q_{y^+}(\lambda),\ \forall \lambda > 0. \tag{8}$$

Note that $Q_{x^-}(\lambda)$ is the proportion of the population which lies below the median by a proportional distance λ or greater. Alternatively, $Q_{x^-}(1-\zeta)$ is the proportion of the population whose income is lower than ζ times the median – this is a frequently-used relative poverty measure. Note that (7) can then be equivalently written as

$$Q_{x^-}(1-\zeta) \geq Q_{y^-}(1-\zeta),\ \forall \zeta \in [0,1]. \tag{9}$$

Hence, one of the necessary conditions for first-order bi-polarization dominance is that relative poverty be uniformly higher in x than in y for all proportions of the median between 0 and 1.

Another simple polarization index is the share of the population within a certain sub-interval of income spread symmetrically on each side of the median income. This is given by $1-Q_x(\lambda)$, where λ is that symmetric spread expressed as a proportion of the median – a popular measure of relative dispersion. $1-Q_x(1)$, for instance, is the proportion of the population located within one median distance of the median.

Note finally that condition (6) can also be rewritten as

$$Q_x(\lambda) \geq Q_y(\lambda),\ \forall \lambda > 0, \tag{10}$$

or as

$$1 - Q_x(\lambda) \leq 1 - Q_y(\lambda),\ \forall \lambda > 0. \tag{11}$$

A symmetric bi-polarization ordering is then obtained when the share of the population away from the median is greater in x than in y, whatever the proportional spread considered, or when the share of the population within the median is lower in x than in y, whatever the spread considered[4].

[4]Note that the popular interquartile range (expressed as a proportion of the median) is given as $d_{x^-}(n/4) + d_{x^+}(n/4)$ whenever $n/4$ is an integer. But neither condition (4) nor condition (6) can be rewritten in terms of that range.

4 Illustration

We use Luxembourg Income Study (LIS) data to illustrate the above tools[5]. Figure 2 shows the distance curves $d(i = np)$ across percentiles p and Figures 3 and 4 display polarization indices $Q_{-}(1-\zeta)$ and $Q(\lambda)$, all computed for the United States (2000) - US00 - the United Kingdom (1999) - UC99 - Canada (1998) - CN98 - the Netherlands (1994) - NL94 - Mexico (1998) - MX98 - and France (1994) - FR94. Figure 2 shows that many pairs of countries can be distinguished – *inter alia*, many of those involving Mexico, the US, France, Canada and the UK. Figure 3 shows why a statistical test of relative poverty can rank unambiguously all possible pairs of countries that do not include the Netherlands – except for Canada's curve which crosses the UK's and which a conventional test of size 5% cannot distinguish statistically for larger values of ζ. Note also that Mexico has the highest level of relative poverty of all 6 countries. Finally, once we impose the symmetry axiom, most pairs of countries are ranked with statistical significance (see Figure 4) and that, in particular, the Netherlands has lower first-order polarization than any one of the other countries (except possibly France).

5 Conclusion

It is well known that "inequalities can diverge" (*e.g.*, Wolfson (1994) and Esteban and Ray (1994)), namely that polarization and inequality can evolve in opposite directions. The distinction between inequality and bi-polarization is even sharper in this paper. This is mainly because of the monotonicity axiom 3 by which movements of income away from the median must increase bi-polarization, regardless of their impact on the mean – a crucial element in accounting for the movements of the usual relative inequality indices. This focus on first-order ethical properties has the advantage of making the bi-polarization comparisons potentially more general. Interestingly, illustrations using cross-country data suggest that a number of important bi-polarization orderings can still be made in spite of this increased ethical generality.

[5] `http://lissy.ceps.lu` for detailed information on the structure of these data. Living standards are measured by household disposable income divided by the squared root of household size. Observations with negative incomes are removed as well as those with incomes exceeding 50 times the median. Household observations are weighted by the LIS sample weights times the number of persons in the household.

A Proof of Theorem 1

Note first that Axiom 1 implies that

$$P\left(d_x\right)=P\left(d_x/m_x\right) \tag{12}$$

for any $x \in \Re^n_{++}$. We can therefore suppose that, for all of the comparisons of bi-polarization that need to be consistent with Axiom 1, all incomes have been pre-normalized by the median of their distribution. We then check in turn the necessity and sufficiency of condition 4:

1. Necessity: $P\left(d_x\right) \geq P\left(d_y\right)$, for all $P \in \mathcal{C}(\mathrm{A1}, \mathrm{A2}, \mathrm{A3})$, implies $d_x(i) \geq d_y(i)$, for all $i = 1, ..., n$.

   Suppose we have $d_x(i) \leq d_y(i)$ for all $i = 1, ..., n$ and $d_x(i) < d_y(i)$ for some $i = 1, ..., n$. Note that

   $$Q_x(d_y(i)) < Q_y(d_y(i)) \tag{13}$$

   But $Q(d_y(i))$ does belong to $\mathcal{C}(A1, A2, A3)$. Hence, we cannot have $d_x(i) < d_y(i)$ anywhere if $P\left(d_x\right) \geq P\left(d_y\right) \forall P(\cdot) \in \mathcal{C}(\mathrm{A1}, \mathrm{A2}, \mathrm{A3})$.

2. Sufficiency: $d_x(i) \geq d_y(i)$, for all $i = 1, ..., n$, implies $P\left(d_x\right) \geq P\left(d_y\right)$, for all $P \in \mathcal{C}(\mathrm{A1}, \mathrm{A2}, \mathrm{A3})$.

   This is straightforward since, by Axiom 3 and whenever $d_x(i) \geq d_y(i) \forall i = 1, ..., n$, we can write:

   $$\begin{aligned} P(d_x(1), d_x(2), \ldots, d_x(n)) &\geq P(d_y(1), d_x(2), \ldots, d_x(n)) && (14)\\ &\geq \ldots && (15)\\ &\geq P(d_y(1), d_y(2), \ldots, d_y(n)) && (16) \end{aligned}$$

   for all of the bi-polarization indices that belong to $\mathcal{C}(A1, A2, A3)$.

B The sampling distribution of the polarization estimators

First, consider the "spreads" $d_x(np)$, where p is some percentile. These spreads are functions of p-quantiles and of the median, the sampling distribution of which was derived by Bahadur (1966). Suppose that a population is characterized by a twice differentiable distribution function F. Then, if the $p-$quantile of F is denoted by $X(p)$, and the sample $p-$quantile from a sample of n independent drawings y_i from F by $\hat{X}(p)$, we have

$$\hat{X}(p) - X(p) = -\frac{1}{nf\big(X(p)\big)} \sum_{i=1}^{n} \Big(I\big(x_i < X(p)\big) - p \Big) + O\big(n^{-3/4} (\log n)^{3/4}\big), \tag{17}$$

where $f = F'$ is the density. The asymptotic sampling distribution of $d_x(np)$ can then be obtained by applying Rao (1973)'s "delta" method.

Second, consider the following estimator $\hat{Q}(\lambda)$ of the bi-polarity indicator $Q(\lambda)$:

$$\hat{Q}(\lambda) = \int_0^{\hat{m}(1-\lambda)} d\hat{F}(y) + \int_{\hat{m}(1+\lambda)}^{\infty} d\hat{F}(y), \tag{18}$$

where $\hat{F}$ is the empirical distribution function. $\hat{Q}(\lambda)$ can be expressed asymptotically as

$$\begin{aligned} \hat{Q}(\lambda) \simeq \quad & (\hat{m} - m) \left[(1-\lambda) f\left(m\left(1-\lambda\right)\right) + (1+\lambda) f\left(m\left(1+\lambda\right)\right) \right] \\ & + \hat{F}\left(m\left(1-\lambda\right)\right) + 1 - \hat{F}\left(m\left(1+\lambda\right)\right). \end{aligned}$$

Everything above can be expressed as sums of iid variables: for $\hat{m}$, see (17), and note that $\hat{F}(y) = \frac{1}{n}\sum_{i=1}^{n} I(x_i \leq y)$. The asymptotic sampling distribution of $\hat{Q}(\lambda)$ then follows by simple computation.

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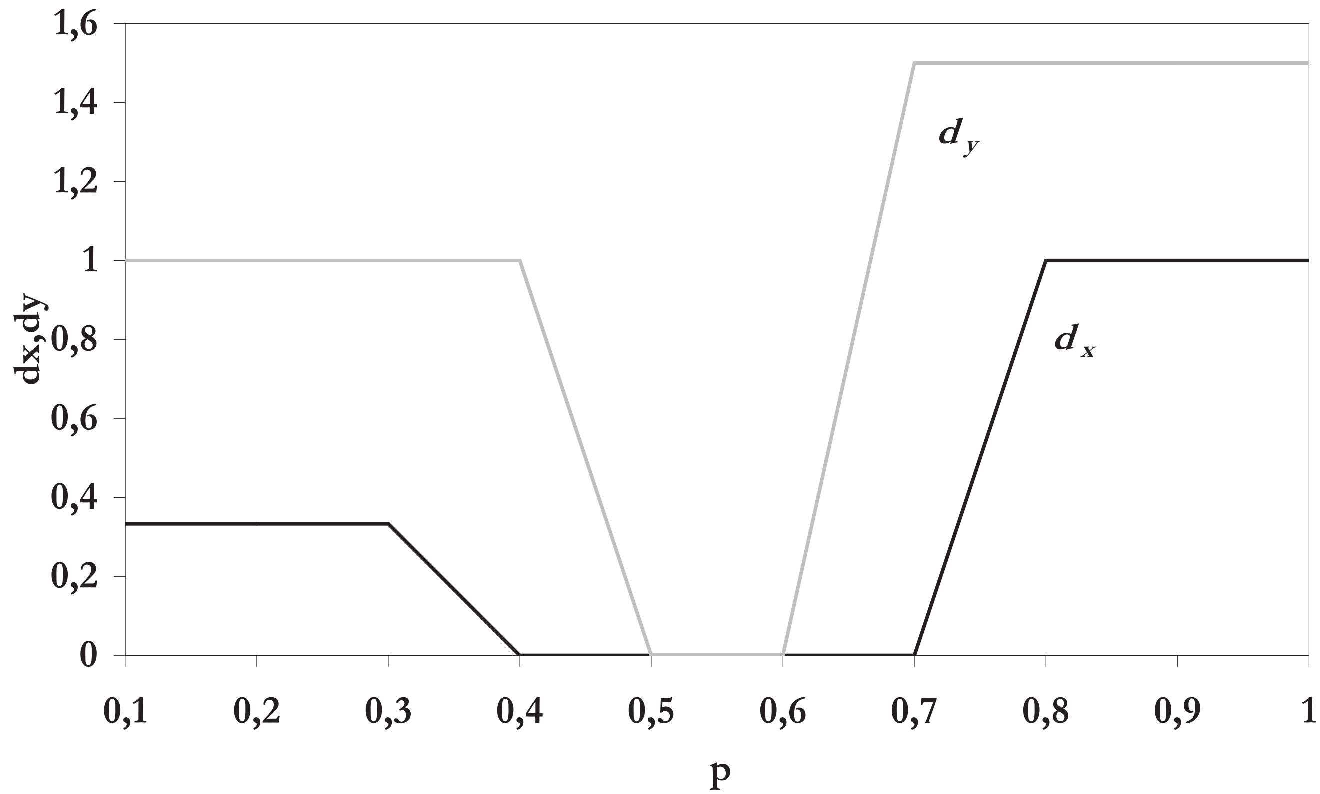

Figure 1: TEST OF FIRST-ORDER BI-POLARIZATION

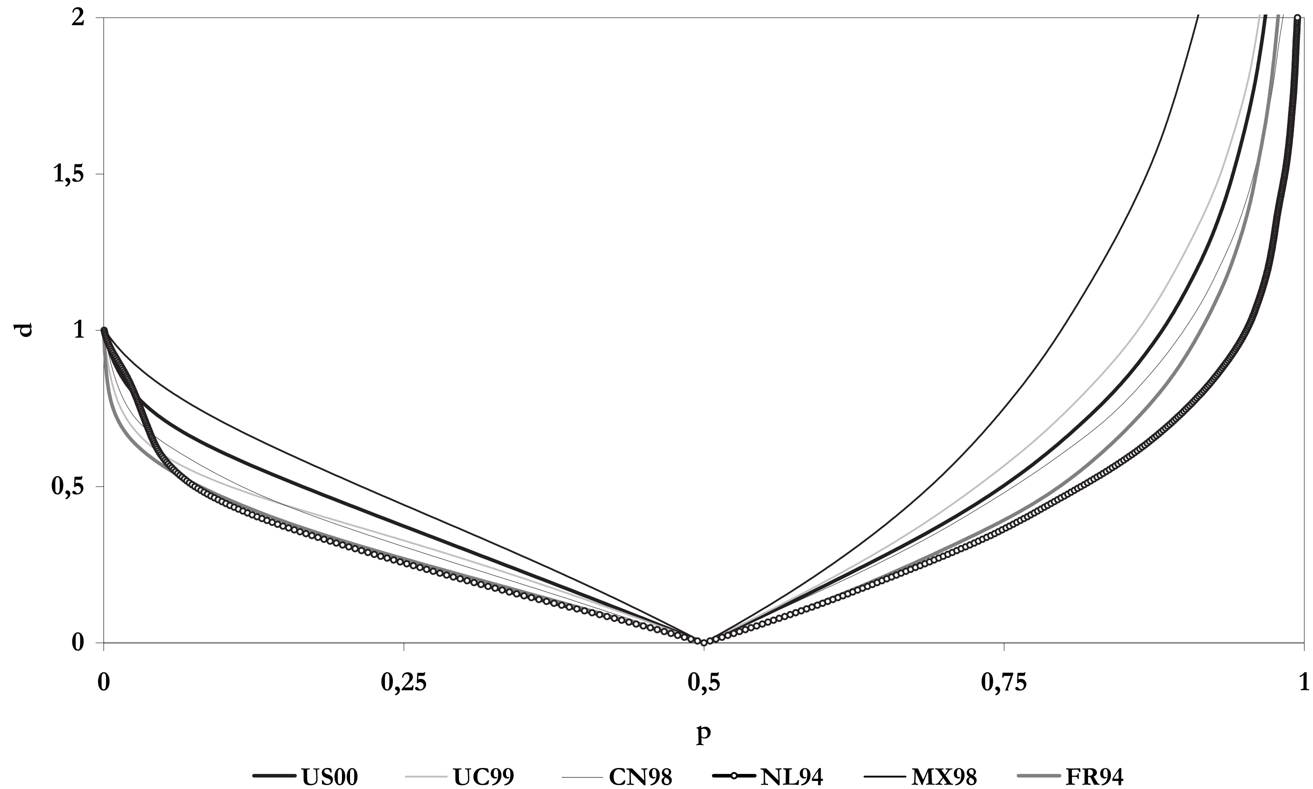

Figure 2: DISTANCE CURVES AT PERCENTILES p $(d(i = np))$

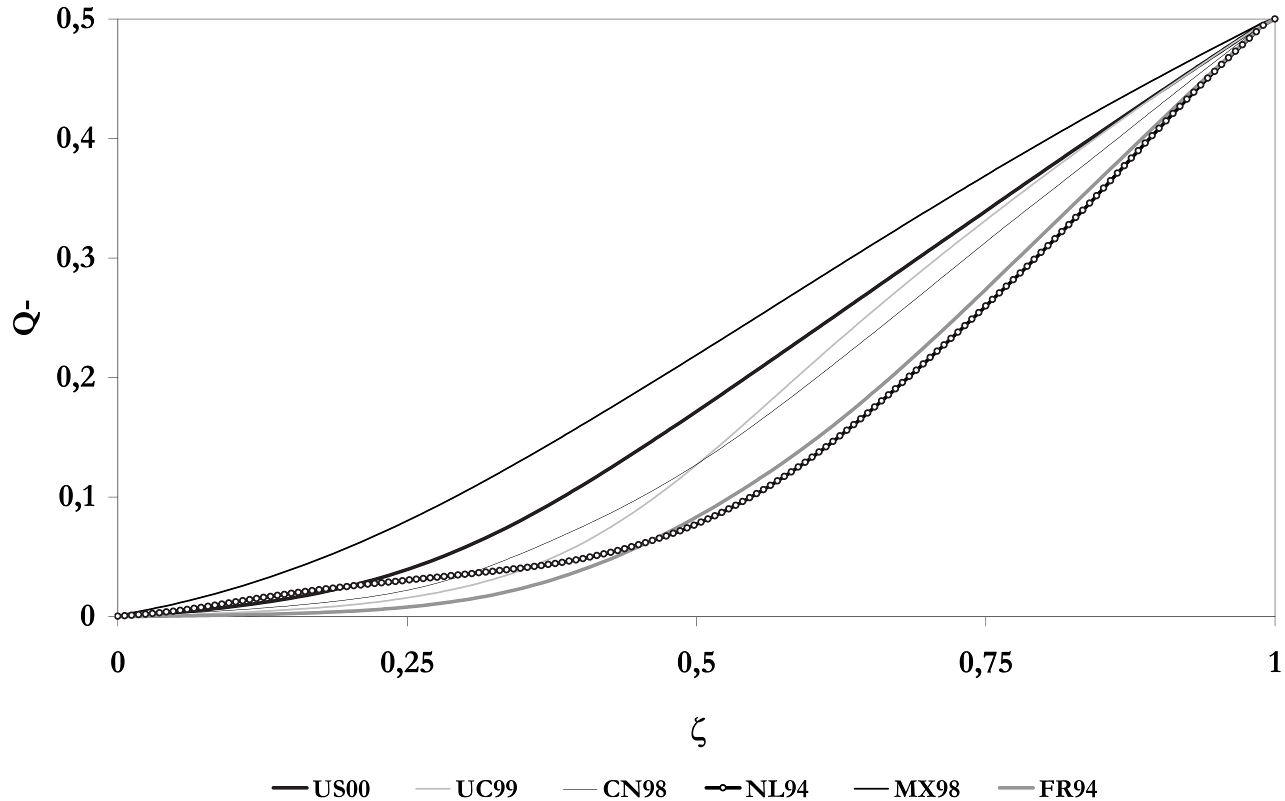

Figure 3: $Q_{-}(1-\zeta)$ INDICES

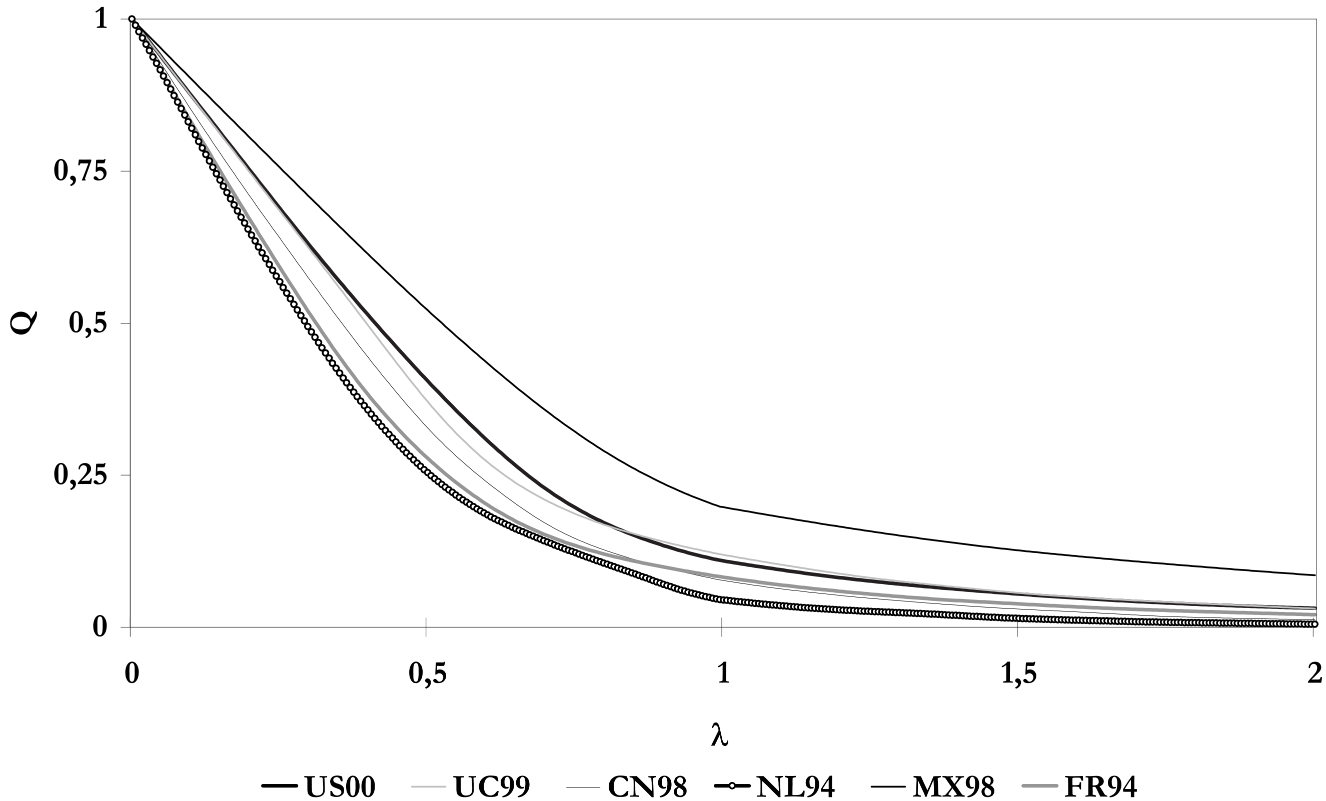

Figure 4: $Q(\lambda)$ INDICES